Misconceptions in mathematics and diagnostic teaching
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Section 1

Introduction
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Assumption 1: Constructivist learning theory

- A pupil does not passively receive knowledge from the environment - it is problematic for knowledge to be transferred faithfully from one person to another.

- A pupil is an active participant in the construction of his/her own mathematical knowledge. The construction activity involves the reception of new ideas and the interaction of these with the pupils extant ideas.
Misconceptions arise frequently because a pupil is an active participant in the construction of his/her own mathematical knowledge via the reception and the interaction of new ideas within the pupils extant ideas.

Extant idea: If I multiply two whole numbers I get a bigger number

New Concept: Decimal numbers and fractions

Misconception: If I multiply two fractions I will always get a bigger number
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Assumptions 3:

1) Teaching is more effective when misconceptions are identified, challenged, and ameliorated.

2) Pupils face internal cognitive distress when some external idea, process, or rule conflicts with their existing mental schema.

We accept the research evidence which suggests that the resolutions of these cognitive conflicts through discussion leads to effective learning.
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Section 2

The difference between a mistake and a misconception in mathematics
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Misconceptions in mathematics are not new; accomplished mathematicians also have misconceptions.

The Renaissance mathematician John Wallis used a naïve method of induction as follows.

He knew that

\[ \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} \]

He also knew the concept of ordinal numbers. So he (mis) applied this to all unitary fractions to obtain:

\[ \ldots < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} < \frac{1}{0} < \frac{1}{-1} < \frac{1}{-2} < \frac{1}{-3} < \ldots \]

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The difference between a mistake and a misconception.

The pupil has shown that s/he i) understands the method to find the solution ii) knows how to open brackets but carelessly makes an error in the 2\textsuperscript{nd} bracket expansion: adding rather than multiplying.

1+5=6 instead of 1×5=5
10-3=7 instead of 10×-3=-30
The pupil understands an algorithm but there is a computational error due to carelessness. Instead of using a calculator, as allowed in the examination, the pupil has, without due care, used mental methods.
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Example 1 of a misconception.

A common pupil misconception: The pupil has transferred the algorithm for multiplying fractions to adding fractions.

\[
\frac{3}{5} + \frac{2}{3} = \frac{5}{8}
\]
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Example 2 of a misconception.

The pupil has misapplied place value to interpret the conjunction of a number and a letter in algebra.

If $x = 5$, $2x$ has been interpreted to be 25.

This is frequently observed.

Source of pupil’s work: QCA
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The salient distinction between a mistake and a misconception.

- **Mistakes**: The pupil understands an algorithm but there is a computational error due to carelessness. A mistake is normally a one-off phenomenon.

- **Misconceptions**: The pupil has misleading ideas or misapplies concepts or algorithms. A misconception is frequently observed.
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Why is the consideration of misconceptions important?

- Children construct meaning internally by accommodating new concepts within their existing mental frameworks.

- Thus, unless there is intervention, there is likelihood that the pupil’s conception may deviate from the intended one.

- Pupils are known to misapply algorithms and rules in domains where they are inapplicable.

- A surprisingly large proportion of pupils share the same misconceptions.
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A pupil says this is the rule to divide two fractions:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \div \frac{c}{d}
\]

a) This is a misconception
b) This is a perfectly good rule
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Some misconceptions in algebra
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\[ x \] is a ‘specific unknown’ misconception.

The pupil here has misapplied the notion of letters as ‘specific unknown’.

Here \( x, y, \) and \( n \) are indeed unknown variables – the pupil decides that, as such, she or he can make them each equal to a convenient number.

A piece of rope 5 metres long is cut into two pieces.
One piece is \( x \) metres long.
How long is the other piece?

\[ 7.5 \text{ metres} \]

There are 24 hours in one day.
How many hours are there in \( y \) days?

\[ 24 \times 3 = 72 \text{ hours} \]

It costs £140 to hire a coach.
This cost is shared equally among \( n \) people.
How much does each person pay?

\[ \frac{140}{n} \text{ each} \]
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A cancelling/deletion misconception.

The unknown \( x \) is in the ratio 1:2 and the pupil misapplies simplifying ratio into this domain: s/he divides the coefficients of \( x \) by 3

Q 1. Solve \( 3x + 3 = 6x + 1 \)

A. \( 3x + 3 = 6x + 1 \)
\[ \Rightarrow x + 3 = 2x + 1 \]
\[ \Rightarrow 2 = x \]

This misconception is also evident in this ‘simplification’.

\[ \frac{\frac{3}{\frac{3}{1} x + 3}}{\frac{2}{1} x + 1} = \frac{3x + 3}{x + 1} \]

Beware: The pupil could argue that it is correct and with justification!
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A misconception with inverses.

The two arithmetic operations are:
*First*: Multiply by 3
*Second*: Add 6

The pupil inverts both to obtain the answer:
*First*: Divide by 3
*Second*: Subtract 6

This misconception can be countered by the ‘socks and shoes’ analogy.

Q. I multiply a number $x$ by 3 and add 6 to obtain 27? What is the number $x$?

A. 27 divided by 3 = 9
   And 9 take away 6 = 3
   So $x = 3$
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Misconceptions about *distributivity* of operations.

In KS3 and KS4, pupils are introduced to the algebraic analogue of the distributive law of arithmetic. For example,

\[ 2(a + b) = 2a + 2b \]

Then there follows the risk of *over-generalising* the rule to operations that are *not* distributive.

\[ (a + b)^2 = a^2 + b^2 \]
\[ \sqrt{a + b} = \sqrt{a} + \sqrt{b} \]
\[ 3(xyz) = 3x3y3z \]
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Equating misconceptions due to overgeneralizing numbers.

The pupil was initially introduced to quadratic equations by investigating equation such as $x^2 - 4x + 3 = 0$.

Solvable in this manner:

$x^2 - 4x + 3 = 0$

$\Rightarrow (x - 3)(x - 1) = 0$ ........(1)

$\Rightarrow (x - 3) = 0$ or $(x - 1) = 0$ ...........(2)

So $x = 3$ or $x = 1$.

The pupil misapplies the method to quadratic equations not equal to zero.

The reasons why (1) leads to (2) needs to be clearly understood to avoid this misconception.

Q. Solve $x^2 - 4x + 3 = 12$

A. $x^2 - 4x + 3 = 12$

A. $(x - 3)(x - 1) = 12$

B. $(x - 3) = 12$ or $(x - 1) = 12$

C. $x = 15$ or $x = 13$
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- All functions are linear misconception

- If \( \sin(angle) = \frac{1}{2} \) then angle = ______?

- Vishen assumed linearity and reasoned as follows: “The angle is 45 degrees ... Because when the angle increases to 90 degrees, it is 1. Half of 90 degrees is 45 degrees.”
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Diagnostic teaching
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Implications of Constructivist learning theory.

1) Because misconceptions arise frequently when a pupil’s intellect engages in the interaction of new ideas with his/her extant ideas, the analysis of misconceptions is crucially important to teaching and learning.

2) Unless a misconception is identified and ameliorated there is the risk of cognitive conflict and/or further misconceptions:

**Misconception:** If I multiply two fractions I will always get a bigger number

**Cognitive conflict:** Find what number multiplied by $\frac{1}{2}$ makes $\frac{1}{4}$
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**Diagnosis of misconceptions.**

- A teacher cannot correct a misconception unless s/he understands the reasons behind it. That is, the teacher has to **diagnose** the faulty interaction between the pupils extant ideas and the new concept by **discussion**.

- Once the **diagnosis** is made then the teacher can challenge or contrast the misconception with the faithful conception. Research evidence shows that a pupil is more likely to not only adopt the faithful conception but also retain the correct understanding by this **diagnostic approach**.

**Misconception:** To add two fractions, I add the numerators and the denominators.

\[ \frac{3}{5} + \frac{1}{2} = \frac{4}{7} \]

**Challenge:** If the rule to add two fractions is to add the numerators and the denominators then

\[ \frac{1}{2} + \frac{1}{2} = \frac{2}{4} \text{ same as } \frac{1}{2} \]

*but we know answer should be 1*
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**Diagnostic teaching.**

“Traditionally, the teacher with the textbook explains and demonstrates, while the students imitate; if the student makes mistakes the teacher explains again. This procedure is not effective in preventing ... misconceptions or in removing [them].

*Diagnostic teaching* ..... depends on the student taking much more responsibility for their own understanding, being willing and able to articulate their own lines of thought and to discuss them in the classroom”.

Source: Swann, M : *Gaining diagnostic teaching skills: helping students learn from mistakes and misconceptions*, Shell Centre publications
Two ways to teach...

A ‘Transmission’ view

An individual activity based on watching, listening and imitating until fluency is attained.

‘Connected’, ‘challenging’ view

A collaborative activity in which learners are challenged and arrive at understanding through discussion.

M. Swann, *Improving Learning in Mathematics*, DFES
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Source of cartoon: M. Swann, Improving Learning in Mathematics, DFES
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Template for diagnostic teaching.

- Lessons focus on known, specific difficulties. Rather than posing many questions in one session, it is more effective to focus on a challenging situation or context and encourage a variety of interpretations to emerge, so that students can compare and evaluate them.

- Questions or stimuli are posed or juxtaposed in ways that create a tension or conflict that needs resolving. Contradictions arising from conflicting methods or opinions create awareness that something needs to be reconsidered, and understandings clarified.
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Template for diagnostic teaching.

- Activities provide opportunities for meaningful feedback to the student on his or her interpretations. This does not mean providing superficial information, such as the number of correct or incorrect answers. Feedback is provided by students using and comparing results obtained from alternative methods. This usually involves some form of small group discussion.

- Lessons include time for whole class discussion in which new ideas and concepts are allowed to emerge. This can be a complex business and requires non-judgmental sensitivity on the part of the teacher so that students are encouraged to share tentative ideas in a non-threatening environment.
A pupil says this is the rule to divide two fractions:

\[
\frac{a}{b} \div \frac{c}{d} = a \div c \div b \div d
\]

Would this pupil conception be a good example for diagnostic teaching?
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Example of dealing with a misconception.

After discussion with a pupil holding a cancelling misconception exhibited opposite you could find that it is based on a misapplication of simplifying ratio (the x coefficients are in the ratio 1:3).

One way to contrast or challenge this is to specialise the example with numbers – say with $x = 2$ – and show that the resulting equality is incorrect.

\[
\frac{3}{6x + 2} = \frac{3x + 2}{2x + 1} \quad \frac{6 \times 2 + 2}{2 \times 2 + 1} = \frac{3 \times 2 + 2}{2 + 1}
\]

\[
\Rightarrow \frac{14}{5} = \frac{8}{3} \Rightarrow 2.8 = 2.666......
\]
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Example 2 of dealing with a misconception.

This pupil likely holds the misconception that a larger area implies a larger perimeter … probably based on the ‘naturalness’ of taking away.

One way to contrast or challenge this misconception that a larger area implies a larger perimeter is to give the pupil examples where it is not true. E.g. 2×2 square and a 1×3 rectangle. Alternatively show that any staircase D made out of rectangle C has perimeter equal to C by a counting exercise.

Source: DfES
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**SUMMARY:**

*Importance of dealing with misconceptions*

1) Teaching is more effective when misconceptions are identified, challenged, and ameliorated.

2) Pupils face internal cognitive distress when some external idea, process, or rule conflicts with their existing mental schema.

3) Research evidence suggests that the resolutions of these cognitive conflicts through discussion leads to effective learning.
Dealing with misconceptions:

**Diagnosis:** Get pupils to explain how they came to their answers or rules.

**Amelioration:** If there is a misconception, challenge it or contrast it with the faithful conception.

“Erroneous conceptions are so stable because they are not always incorrect. A conception that fails all the time cannot persist. It is because there is a local consistency and a local efficiency in a limited area, that these incorrect conceptions have stability.”