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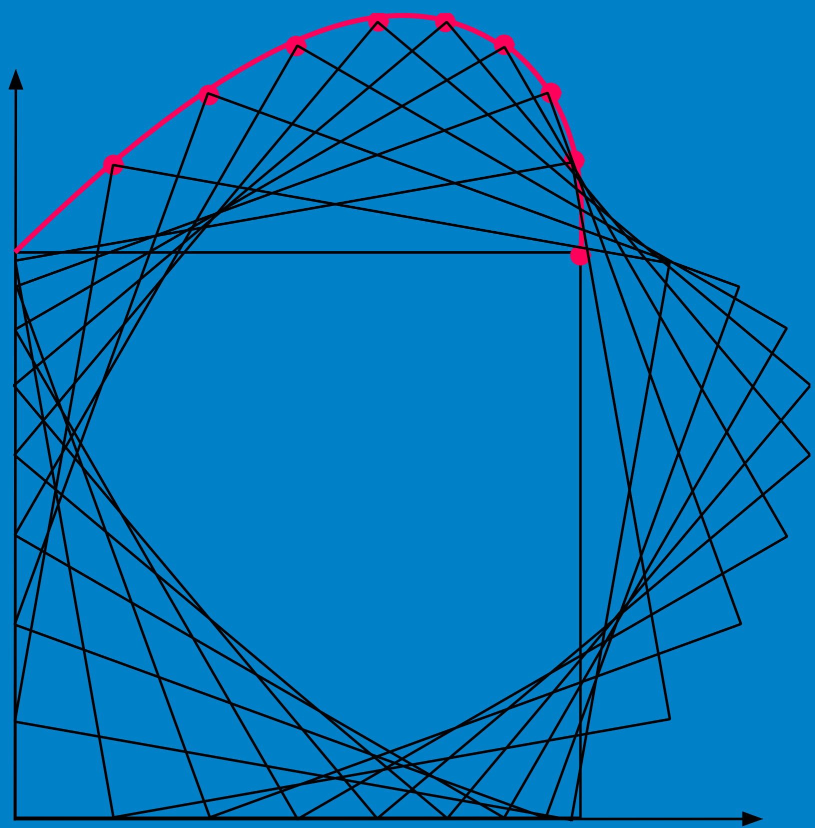

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Articles submitted will be reviewed by the editors and members of the Editorial Board. The Board will ensure that the papers make a contribution to our understanding of mathematics learning and teaching, that the mathematics presented is correct, and that the language and layout used is user friendly. Support will be provided by the editors to contributors in relation to meeting the above requirements.

The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.

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From the Editor

Dear LTM readers

In the first article of LTM 29, Duncan Samson explores an interesting classroom investigation that involves finding polygons with numerically equal area and perimeter. In the second article in this issue, Craig Pournara shares examples of the worksheets developed by the Wits Maths Connect Secondary team, and gives us a glimpse into the rationales and strategies behind the design of the various tasks. The third article, by Yiu Suen Tang and Kin Keung Poon, describes a number of GeoGebra applets that have specifically been designed to allow students to engage with fractions in a dynamic way, while in the fourth article JP le Roux reflects on assessment during the COVID-19 pandemic and encourages us to think critically about the symbiotic relationship between assessment and learning. James Metz then explores the locus of a vertex of squares and rectangles being rotated under specific constraints.

In the sixth article, Harry Wiggins showcases the usefulness of logarithms in the context of large numbers, while in the seventh article Yiu-Kwong Man presents a simple visual approach to deriving the quadratic formula. Michael de Villiers then demonstrates the usefulness of using directed or signed quantities in geometry, after which Alan Christison explores an interesting probability problem by making use of linear constraints to create a feasible region. In the tenth article, Michael de Villiers presents and proves a number of theorems relating to tangential or circumscribed quadrilaterals, after which Poobhalan Pillay's book "1000 Mathematics Olympiad Problems" is reviewed.

We hope you enjoy the diverse array of articles in this issue, and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, can be found at the end of this journal. If you have an idea but aren't sure how to structure it into an article, you are welcome to email the editor directly – we'd be happy to engage with you about turning your idea into a printed article.

Duncan Samson

Polygons with Numerically Equal Area and Perimeter

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INTRODUCTION

An interesting classroom investigation involves finding polygons with numerically equal area and perimeter. Perhaps the simplest example of such a scenario is a square with side length 4 units which has a perimeter of 16 units and an area of 16 units^2. Another classic example is a 5-12-13 right-angled triangle which has a perimeter of 30 units and an area of 30 units^2. In this article I take these two examples as starting points and expand the discussion into what I hope could be emulated in the classroom as an investigation or guided exploration.

EXPLORING SQUARES

Other than a square of side length 4 units, are there any other squares that have the property that their area and perimeter are numerically equal? Perhaps a good starting point is simply to tabulate area and perimeter for a variety of different squares. Restricting ourselves to integer side lengths for the moment, we could set up a table like this:

Side length (u)	Perimeter (u)	Area (u^2)
1	4	1
2	8	4
3	12	9
4	16	16
5	20	25
6	24	36
7	28	49
8	32	64
9	36	81

TABLE 1: Table of areas and perimeters for squares of different side length

A nice way to do this, particularly if one wanted to include fractional side lengths, is to set up a spreadsheet and enter cell formulas for perimeter and area and then simply copy the formulas down for different side lengths. Even from this short list pupils should hopefully get a sense that once the side length is greater than 4 units the area increases far more rapidly than the perimeter, and the two will never be the same value again. For a side length less than 4 units, pupils should also have a sense that the area is always going to be smaller than the perimeter. A side length of 4 units thus seems to be the only scenario for a square having numerically equal area and perimeter.

Let's see if we can formalise what we have an intuitive sense of from Table 1. For a square with side length x, the perimeter and area will be $4x$ and x^2 respectively. If the area and perimeter are numerically equal then $x^2 = 4x$. This is a quadratic equation which we can rearrange and factorise as follows:

$$x(x-4) = 0$$

The two solutions to this quadratic equation are $x = 0$ and $x = 4$, and since x represents the side length of the square we can ignore the solution $x = 0$, thus confirming our hunch that a side length of 4 units is the only possible solution to the problem.

We could explore this visually by plotting graphs of $y = 4x$ and $y = x^2$ for $x \geq 0$. The visual also confirms our suspicions.

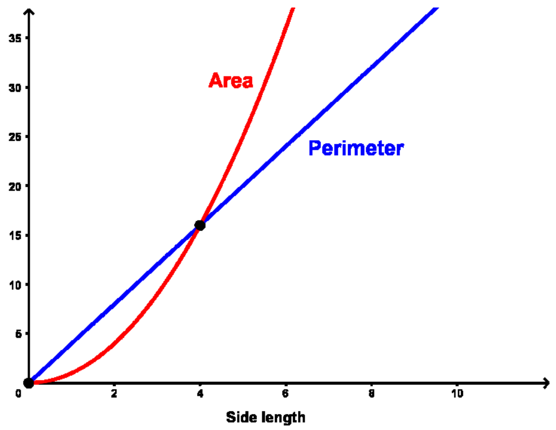

FIGURE 1: Exploring the relation between area and perimeter graphically

EXPLORING RECTANGLES

Now that we have identified a square of side length 4 units as having numerically equal area and perimeter, one might perhaps wonder about rectangles more generally. Are there any other rectangles with integer side lengths that have this property? Let's imagine a rectangle with side lengths x and y. If the area and perimeter are numerically equal then the equation we need to solve is:

$$xy = 2x + 2y$$

Rearranging and making y the subject we have:

$$xy - 2y = 2x$$
$$y(x-2) = 2x$$
$$\therefore\ y = \frac{2x}{x-2}$$

For integer side lengths we simply need to find integer solutions for both x and y. To start with, it is worth noting that since both x and y are positive, from the above we have the restriction $x > 2$. By substituting in integer values for x we generate the following values for y:

x	3	4	5	6	7	8	9	10
$y = \frac{2x}{x-2}$	6	4	3,333…	3	2,8	2,666…	2,571…	2,5

TABLE 2: Table of related y-values for integer values of x.

In addition to a square with side length 4, the table also reveals a further example – a rectangle with side lengths of 3 units and 6 units, which would have a perimeter of 18 units and an area of 18 units^2. But are there any others?

Notice that since $x > 2$, it will always be true that $2x > x - 2$. This means that as x increases, y will decrease. From Table 2 it looks as though the rate at which y is decreasing is slowing down. One can confirm this, and even get a sense that y will never drop below a value of 2, by setting up a spreadsheet and entering x-values up to 100 for example. If we explore this more algebraically, then we can manipulate the expression for y as follows:

$$y = \frac{2x}{x-2} = \frac{2x-4+4}{x-2} = \frac{2(x-2)+4}{x-2} = 2 + \frac{4}{x-2}$$

Notice that as x increases, the fraction $\frac{4}{x-2}$ gets smaller and smaller and the value of y gets closer and closer to 2. Expressed more formally, as $x \to \infty$, $y \to 2$. This confirms our hunch based on the table.

Using a graphical approach, note that $y = \frac{4}{x-2} + 2$ is simply a hyperbola (Figure 2). The three solutions identified in Table 2 are indicated on the graph. Since there are no integer value of x for $x \in (2; 3)$ and since there are no integer values of y for $y \in (2; 3)$, this confirms that the only two rectangles with integer side lengths that have numerically equal area and perimeter are the 3-by-6 rectangle and the 4-by-4 square. Note however that if we don't restrict ourselves to integer side lengths then every point on the curve represents a solution.

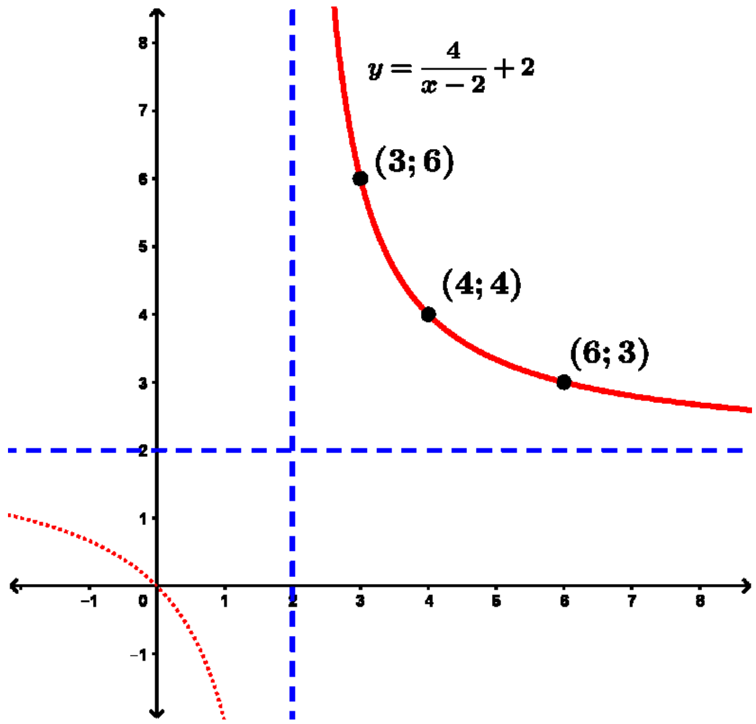

FIGURE 2: Graph of $y = \frac{4}{x-2} + 2$ for $x > 2$

EXPLORING TRIANGLES

As mentioned in the introduction, another classic example of a polygon with numerically equal area and perimeter is a 5-12-13 right-angled triangle, which has a perimeter of 30 units and an area of 30 units^2. Are there any other right-angled triangles, with integer side lengths, that have this property? Let's imagine a right-angled triangle with perpendicular side lengths a and b. The hypotenuse of this triangle would be $\sqrt{a^2+b^2}$, and its area would be $\frac{1}{2}ab$. If the area and perimeter were numerically equal, then we would need to find integer solutions to the equation:

$$a+b+\sqrt{a^2+b^2}=\frac{1}{2}ab$$

While doable, this is going to be a somewhat cumbersome process. Interested readers are nonetheless encouraged to have a go. A slightly more elegant approach to the problem is to make use of the fact that Pythagorean triples a, b, c can be generated from:

$$a=m^2-n^2 \quad ; \quad b=2mn \quad ; \quad c=m^2+n^2$$

where m and n are positive integers with $m>n$. We thus have:

$$\text{Perimeter}=a+b+c=2m^2+2mn$$

$$\text{Area}=\frac{1}{2}ab=mn(m^2-n^2)$$

If the area and perimeter were numerically equal, then we would need to find integer solutions to the equation:

$$2m^2+2mn=mn(m^2-n^2)$$

$$\therefore \quad 2m(m+n)=mn(m+n)(m-n)$$

Since m and n are both positive integers, we can divide through by m and $(m+n)$ since neither can equal zero. Thus:

$$2=n(m-n)$$

This can be rearranged to give:

$$m=n+\frac{2}{n}$$

Now, since m and n are both positive integers, n can only take on values 1 and 2, with an associated value of $m=3$ in both cases. When $n=1$ and $m=3$ the Pythagorean triple a, b, c generated is 8,6, 10. When $n=2$ and $m=3$ the Pythagorean triple a, b, c generated is 5,12,13. These are thus the only right-angled triangles with integer side lengths for which the area and perimeter are numerically equal.

EXPLORING REGULAR POLYGONS

We have identified a 4-by-4 square as having numerically equal area and perimeter. What about other regular polygons? Let us loosen the restriction of integer side lengths and look more generally at the required conditions for a regular polygon to have this property. To begin with, let's consider a regular pentagon with side length x.

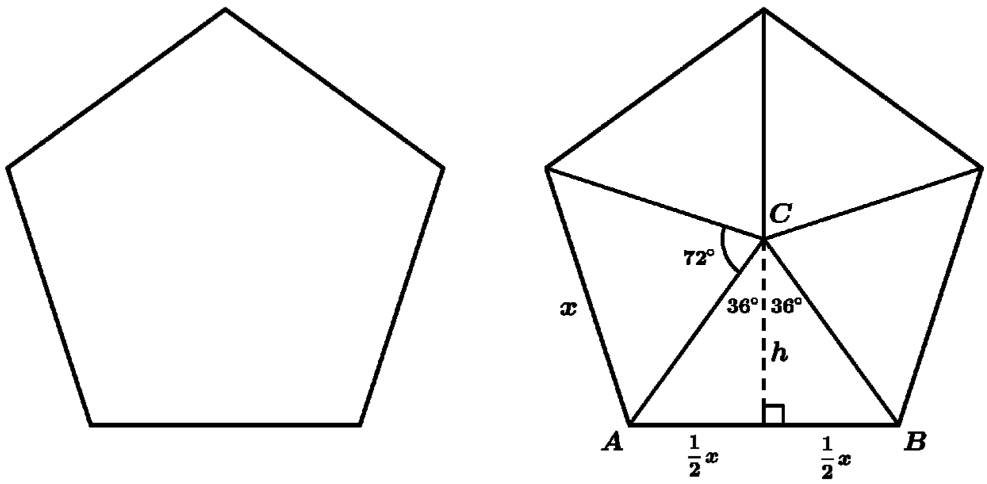

FIGURE 3: Regular pentagon with side length x

If we subdivide the pentagon into five identical isosceles triangles, then the area of each of these triangles is:

$$Area\ \Delta ABC = \frac{1}{2}(x)(h)$$

Since $\tan 36° = \left(\frac{1}{2}x\right)/h$ we have $h = \left(\frac{1}{2}x\right)/(\tan 36°)$. We can thus write:

$$Area\ \Delta ABC = \frac{1}{2}x \times \frac{\frac{1}{2}x}{\tan 36°} = \frac{x^2}{4\tan 36°}$$

$$\therefore\ Area\ of\ pentagon = \frac{5x^2}{4\tan 36°}$$

If the area and perimeter are numerically equal, then:

$$\frac{5x^2}{4\tan 36°} = 5x$$

We thus have $x = 4\tan 36°$.

If we follow a similar process for a hexagon with side length x, then the perimeter of the hexagon is $6x$ and the area of the hexagon is $\frac{6x^2}{4\tan 30°}$. Equating the two expressions and solving for x gives $x = 4\tan 30°$. If we generalise this to an n-sided regular polygon then the side length for which the perimeter and area are numerically equal is:

$$x = 4\tan\left(\frac{180°}{n}\right)\ \ ;\ \ n \geq 3, n \in \mathbb{Z}$$

If we test this formula for a square (i.e. $n = 4$) then $x = 4\tan\left(\frac{180°}{4}\right) = 4\tan 45° = 4$ as previously obtained. Plotting the graph of $y = 4\tan\left(\frac{180°}{n}\right)$ shows how the side length decreases as the number of sides increases.

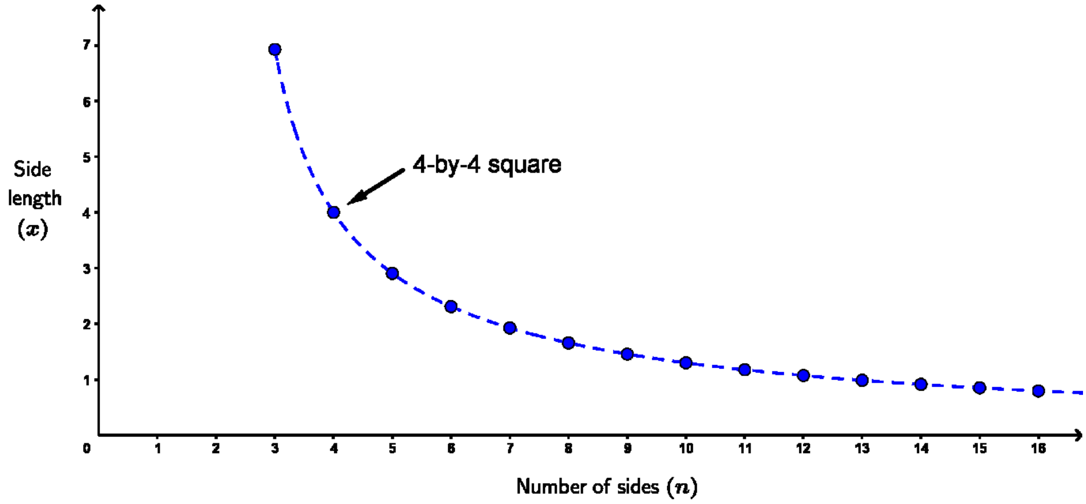

FIGURE 4: Graph of number of sides versus side length

Note that as $n \to \infty$, $\frac{180°}{n} \to 0$ and $\tan\left(\frac{180°}{n}\right) \to 0$. The limiting case is the circle which would have numerically equal area and perimeter when $2\pi r = \pi r^2$, i.e. when the circle has a radius of 2 units. This value of $r = 2$ is not without significance. We have established that for a regular n-sided polygon to have numerically equal area and perimeter the side length must be $x = 4\tan\left(\frac{180°}{n}\right)$. Furthermore, from Figure 3 we see that in general we have:

$$\tan\left(\frac{180°}{n}\right) = \frac{\frac{1}{2}x}{h} \quad \Rightarrow \quad h = \frac{\frac{1}{2}x}{\tan\left(\frac{180°}{n}\right)}$$

$$\therefore \quad h = \frac{\frac{1}{2} \times 4\tan\left(\frac{180°}{n}\right)}{\tan\left(\frac{180°}{n}\right)} = 2$$

The perpendicular height of each isosceles triangle is thus 2, independent of how many sides the polygon has. If one hasn't come across this result before, it is rather surprising. In essence it means that all regular polygons that have the property that their area and perimeter are numerically equal have an inscribed circle with a radius of 2 units.

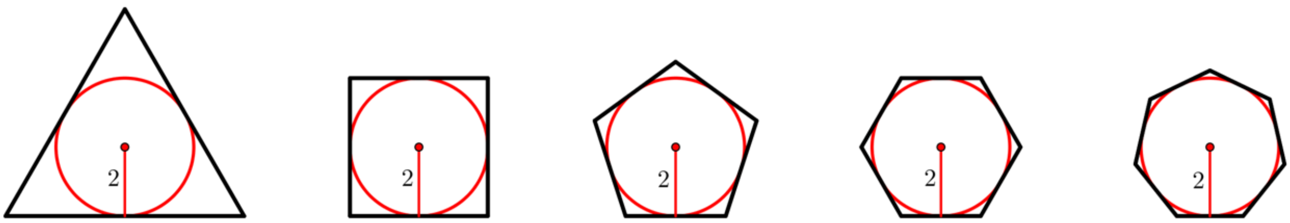

FIGURE 5: Regular polygons with numerically equal area and perimeter

One can arrive at this result in a far more direct way as follows. With reference to Figure 6, note that each triangular subdivision has area $\frac{1}{2}xh$. In general, for an n-sided regular polygon with side length x, the polygon would have area equal to $\frac{1}{2}nxh$ and a perimeter equal to nx. Equating these two expressions yields:

$$\frac{1}{2}nxh = nx$$

And since $n \neq 0$ and $x \neq 0$ we can divide both sides by nx:

$$\therefore \quad \frac{1}{2}h = 1$$

$$\therefore \quad h = 2$$

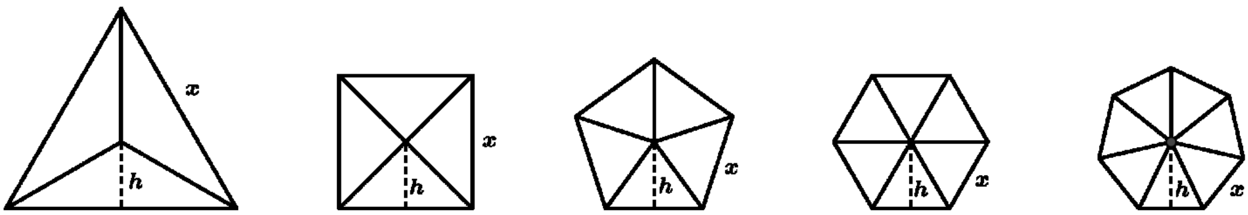

FIGURE 6: Regular n-sided polygons with side length x subdivided into n isosceles triangles

CONCLUDING COMMENTS

Two-dimensional shapes that have numerically equal area and perimeter are popularly known as *equable* or *perfect* shapes. A great deal has been written about equable shapes over the years, and they remain a source of fascination. What I hope I have illustrated in this article is how a simple starting point – the observation that a square with side length of 4 units has numerically equal area and perimeter – can be developed into a classroom investigation suitable for a wide range of age groups and mathematical ability. From an initial process of simply tabulating side length, perimeter and area for different squares, the scope was extended to include other rectangles and right-angled triangles with integer side lengths, and finally regular polygons. In doing so we were able to touch on aspects of number theory, algebraic reasoning, and graphical representations of functions.

There is a great deal more that one could explore in relation to equable shapes, but as a topic for classroom investigation I think what I have included in this article has the potential to encourage rich and meaningful discussion as well as further independent exploration. The amount of scaffolding and guidance pupils will require will vary, and the depth to which one might go would of course depend on the grade of the class. However, since the task is so wonderfully divergent, able pupils can always forge ahead on their own, and this aspect of the investigation is particularly useful for mixed ability classes.

Point Plotting for a Bigger Purpose in Grades 8 and 9

Craig Pournara and the Wits Maths Connect Secondary team[1]

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INTRODUCTION

Plotting points and reading off their coordinates are fundamental skills in high school mathematics. They lay the foundations for learning functions, transformations and analytical geometry. Our research conducted in 2018 showed that only one-third of Grade 9s could write the coordinates of the intercepts when given the graph of a straight line. This suggested that a focus on point plotting is important. As we began developing materials, we realised the range of important mathematical ideas that can be learned through plotting points and working with their coordinates. In fact, this should be the point of working with points in Grades 8 and 9 – not just plotting points as an end in itself. This is why we called our worksheet collection *WhatsThePoint*! In this article we share ideas and examples from the worksheet collection developed by the Wits Maths Connect Secondary (WMCS) project. We provide rationales and strategies behind our task designs to share with teachers what we are trying to bring into focus for learners to learn.

WHAT ARE THE KEY MATHEMATICAL IDEAS?

We believe that working with points can lay a strong mathematical foundation for senior secondary maths, as learners learn to plot, translate and reflect points; to generalise about the effects of moving points; and to identify relationships between the coordinates of points. We draw on the ideas of variation theory in the design of our materials. In doing so, we frequently ask learners to focus on *what changes* and *what stays the same*, and then to consider what they can generalise from their observations. We identified nine key mathematical ideas which can be brought into focus through attention to point plotting. They are:

1. All points in the same vertical line have the same x-coordinate.
2. All points in the same horizontal line have the same y-coordinate.
3. All points on the x-axis have a y-coordinate of zero.
4. All points on the y-axis have an x-coordinate of zero.
5. When you translate a point vertically, the y-coordinate changes but the x-coordinate remains the same.
6. When you translate a point horizontally, the x-coordinate changes but the y-coordinate remains the same.
7. When you reflect a point across the x-axis, the y-coordinate changes but the x-coordinate is unchanged.
8. When you reflect a point across the y-axis, the x-coordinate changes but the y-coordinate is unchanged.
9. If points lie on the same straight line, then they have the same relationship between the x-coordinate and the y-coordinate.

[1] The *WhatsThePoint* materials were developed by members of the WMCS team: Micky Lavery, Wanda Masondo, Vasantha Moodley and Yvonne Sanders, with additional inputs from Hailey Loveland, Sheldon Naidoo, Kate Pournara and Andrew Pournara.

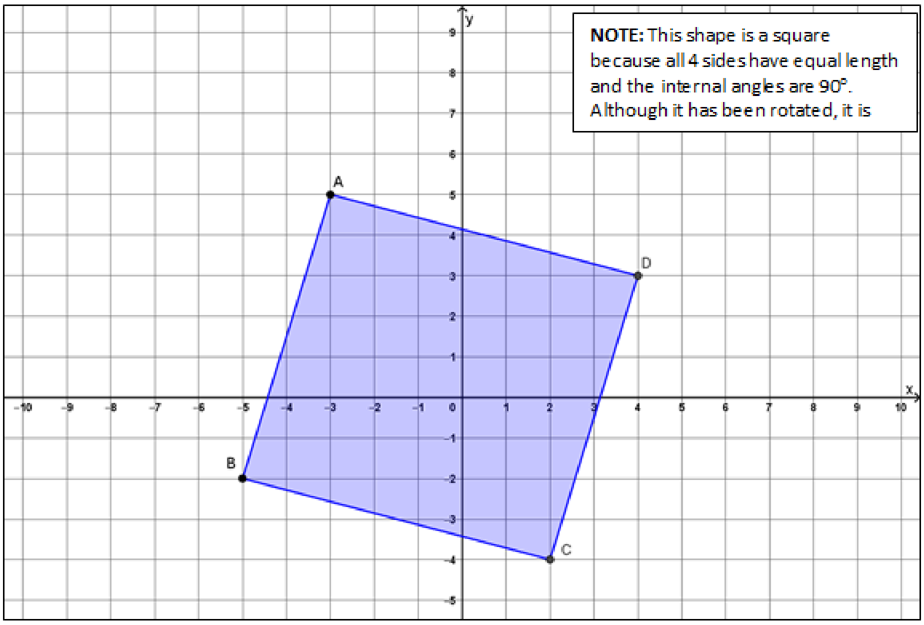
In addition, it is important for learners to make connections between different representations of points such as coordinate pairs, table-representations, graphical representations on the Cartesian plane and even flow/function diagrams. As with all mathematics, the role of logical reasoning cannot be under-estimated. So, many of our tasks require learners to reason about relationships, to justify their thinking and to imagine points and shapes beyond what is available in the diagrams we provide.

DESIGNING THE TASKS

In this section we share four foci of our tasks. Each of these is illustrated with examples from the materials.

FOCUS 1: DEVELOPING LOGICAL REASONING WITH SPATIAL TASKS

1) The diagram shows a square with vertices A, B, C and D.
   a) Write down the coordinates of each vertex.



   b) Points P, Q and R are described below. Plot the points on the Cartesian plane above and label each point clearly.
      i) Point P has the same x-coordinate as A and the same y-coordinate as D.
      ii) Point Q has the same x-coordinate as D and the same y-coordinate as B.
      iii) Point R lies on the same vertical line as B and the same horizontal line as A.

2) You want to move from one point to another but you can only move horizontally and vertically.
   a) Write down the instructions to get from B to D.
   b) Write down the instructions to get from A to point T with coordinates (11;-1).

3) In this question you must identify points that meet the conditions in the middle column. Each point must meet all conditions. Plot each point on the diagram. Write down the coordinates.

Name of point/s	Description of the point	Coordinates of points
E, F	Lies on the x-axis but outside the square	e.g. E(-5;0) F(;)
G, H	Lies on the y-axis and inside the square	e.g. G(0;-2) H (;)
J, K, L	In quadrant II and inside the square	
M, N	On the same horizontal line as B and with positive x-values	
V	Outside the square and very close to vertex C	

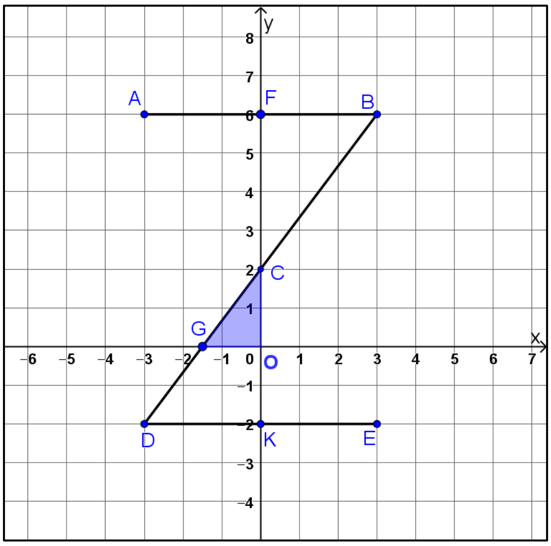
FIGURE 1: Extract from Grade 8 worksheet on points that meet several conditions

Difficulties in spatial reasoning may in part be related to learners not knowing when to separate vertical and horizontal components and when to combine them. In the task shown in Figure 1, we ask learners to give coordinate of points that meet one, two or even three conditions. This requires them to connect the verbal instructions with the diagram, and to pay attention to horizontal and vertical components of points. At the same time, learners are practising plotting points and reading off the coordinates but this practice is in service of another purpose. This is an example of what we call *practice with a purpose*.

FOCUS 2: CONNECTING POINT PLOTTING WITH OTHER TOPICS

Working on the Cartesian plane provides opportunities to make links with other topics in Grades 8 and 9 such as length/distance, area and the Theorem of Pythagoras. This provides good preparation for analytical geometry in Grade 10 and beyond. In the task below, we illustrate ways in which this can be done using a Z-shape.

1) The diagram shows a Z-shape drawn on the Cartesian plane.
   a) Write down the coordinate of points A, B, D and E.
   b) Compare the coordinates of points A and B: what is the same, what is different?
   c) Compare the coordinates of points B and E: what is the same, what is different?
   d) Determine the distance from B to E.
   e) Write down the coordinates of the point that is half-way between B and E.



2) The Z-shape intersects the axes in 4 places.
   a) Give the coordinates of each point.
   b) Explain how we know that the exact position of G is (-1,5;0).

3) Imagine you have to walk along the Z-shape from A to B to D and stop at E. How many units will you walk? (Hint: Think about BD as the hypotenuse of a right-angled triangle).

4) A point T lies 5 units to the left of A, on the same horizontal line.
   a) Write down the coordinates of T.
   b) What is the distance from D to T?

5) Look at ΔGCO in the diagram.
   a) Determine the area of the triangle.
   b) Determine the perimeter of the triangle.
   c) Write down the coordinates of 3 points that lie inside the triangle.

FIGURE 2: Extract of task connecting point plotting to other topics

As with most of our worksheets, we begin with reading off coordinates and comparing similarities and differences. Then we look at relationships between points. In this worksheet we include attention to length, distance, area and an application of the Theorem of Pythagoras (although we provide a very clear hint!). We also push learners to visualise points which lie "outside the frame" of the given figure. While Q5c might seem disconnected from the preceding questions, it reinforces a basic idea that many learners don't seem to appreciate – that the coordinates of points can take on fractional values.

FOCUS 3: FUNCTIONAL RELATIONSHIPS – FROM RULES TO POINT PLOTTING

We know that learners have difficulty in generalising functional relationships – between inputs and outputs, between figural patterns and the figure number, and between x- and y-values in tables. This may be at the heart of their difficulties in finding the equation of a linear function, given the graph. All this relates to point 9 on our list of key mathematical ideas – points that lie on the same line obey the same rule, and conversely, if points don't lie on the line, then they don't obey the rule. While this is a fundamental idea in the topic of functions, it is a not given much attention in the curriculum.

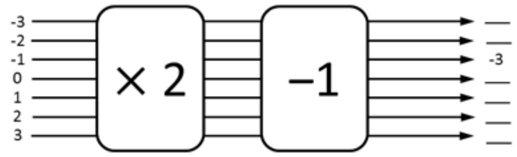
All the points on this worksheet must obey the rule:

To get the output, you must double the input and then subtract 1

e.g. if the input is -1, then the output is: $2(-1) - 1 = -3$

1) Write the rule as an algebraic statement: $y =$ ___________
2) Complete the flow diagram, by calculating the missing output values
3) Complete the table by filling in the y-coordinate of each point.
4) Write down the coordinate pairs for all the points. We have chosen the labels A – G.
5) Plot the points on the Cartesian plane. What do you notice?

Flow diagram



Table

x	-3	-2	-1	0	1	2	3
y					1		

Coordinate pairs

A   B   C   D   E (1;1)   F   G

Graph

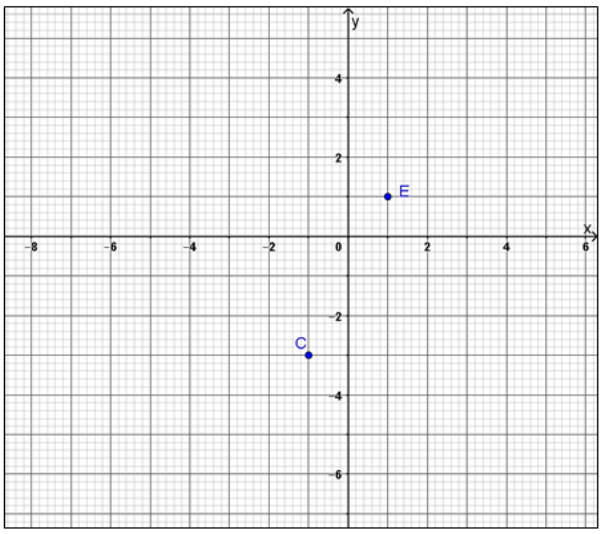
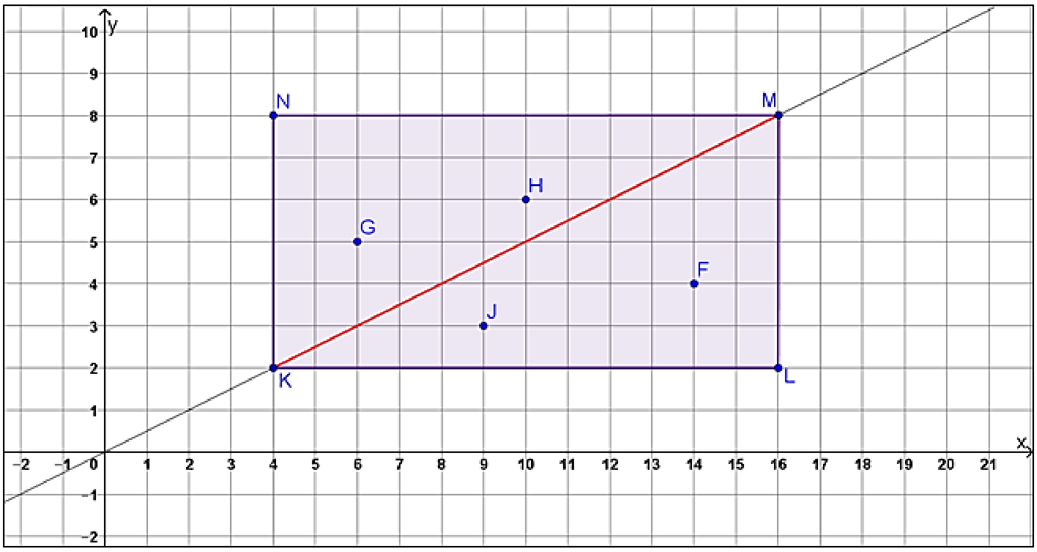


FIGURE 3: Example of tasks promoting connections between verbal rule and other representations

In the Grade 8 task in Figure 3, we focus on one functional relationship in five different representations: verbal description of a rule, algebraic formula, flow diagram, table of values and graph. We do so using a simple rule: "To get the output, you must double the input and then subtract 1". In earlier worksheets, we take the learners slowly through the different representations and make explicit connections. Here we assume the connections are starting to emerge spontaneously for learners, and that ultimately they will see that the verbal relationship can be represented as a straight-line graph which is created by plotting lots of points which obey a given rule.

FOCUS 4: FUNCTIONAL RELATIONSHIPS – FROM POINTS TO RULES

1) Points K, L, M and N form a rectangle which is shown in the diagram.
   a) Write down the coordinates of each point.
   b) Determine the perimeter of the rectangle.



2) The points F, G, H and J lie inside the rectangle. Write down the coordinates of each point in the table.

Original position of point	New position of point
F	F'
G	G'
H	H'
J	J'

3) KM is a diagonal of the rectangle.
   a) Move point F horizontally so that it lies on the diagonal. Label the new point F'. Write down its coordinates in the table.
   b) Move points G, H and J horizontally so that they lie on the diagonal. Label the new points G', H' and J' respectively. Write down the new coordinates of each point in the table.
   c) Look at the relationship between the x-coordinate and the y-coordinate of each new point. Describe the relationship in words.
   d) Points K and M also lie on the diagonal. Look at the x- and y-coordinates of these points. Do they have the same relationship as the new points in the table?

FIGURE 4: Extract from worksheet linking points to the diagonal of a rectangle

As already noted, a key idea in functions is that points lying on the same line obey the same rule. In this worksheet we translate points onto the diagonal of a rectangle and then focus on the relationship between the x- and y-coordinates, noticing that they all obey the same rule. In case of the example in Figure 4, the rule is $y = \frac{1}{2}x$.

In a later section of this worksheet, we push learners to generalise about the relationship, beyond the rectangle itself, and to visualise "outside the frame". For example:

- The diagonal extends outside the rectangle. Give the coordinates of a point that:
  - lies on the diagonal, is outside the rectangle and is in quadrant I
  - lies on the diagonal and is in quadrant III
  - lies above the diagonal and has an x-coordinate larger than 20
  - lies very close to the diagonal but below it and has an x-coordinate larger than 50

There are three worksheets focusing on the above ideas, each getting progressively more challenging. In one worksheet learners explore the impact of moving points horizontally *and* vertically to lie on a given line.

Concluding comments

We trust that we have provided some convincing evidence and examples of the potential that exists in using point-plotting tasks to develop bigger mathematical ideas in Grades 8 and 9. This is particularly important in light of the CAPS curriculum. Even a basic reading of the curriculum documents reveals that the teaching of point plotting is not mentioned in relation to functions. Rather, the introduction to point plotting in Grade 8 is located in the transformation geometry section. While this is not necessarily problematic, the recommended teaching sequence provided by the Department of Basic Education places transformation geometry *after* functions and graphs in Grade 8. This is clearly problematic. In addition, the curriculum trimming to address the impact of the COVID-19 pandemic has led to the removal of transformation geometry in Grade 8 in 2020. In so doing, this has unintentionally removed the introduction to the Cartesian plane and thus to point plotting.

In our materials, we assume learners have been taught the basics of plotting, reading off, translating and reflecting points. Nevertheless, each worksheet begins with reading off the coordinates and/or plotting points, and we provide a succinct summary of the key terminology and ideas in our Grade 8 pack. Our materials are freely available in two downloadable collections, *WhatsThePoint Grade 8* and *WhatsThePoint Grade 9*. Each booklet contains ten worksheets with solutions.

Visit: www.witsmathsconnectsecondary.co.za/resources. Search on "point plotting".

Acknowledgement

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Dynamic Engagement with Fractions using GeoGebra

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INTRODUCTION

The purpose of this article is to share and describe a number of GeoGebra applets that have specifically been designed to allow students to engage with fractions in a dynamic way. The applets can be used to guide students in their conceptual understanding of fractions as they create and compare visual representations of fractions. Both the bar model and the circular model are used so that students become familiar with different representations of fractions early on.

BAR MODEL

In the basic setting of the bar model applet (Figure 1), three rectangular bars are shown. The purpose of this basic setting is to compare and contrast unit fractions. The top bar is fixed and is provided for illustrative purposes. For the other two bars the denominator can be changed by using the sliders. The denominator represents the number of equal parts the rectangular bar is divided into. The numerator remains fixed at 1 so that different unit fractions (shown in pink and blue for their respective bars) can be compared. When the 'show translation' box is ticked the unit fraction from the bottom rectangle can be dragged upwards (using the slider on the left) so that it overlaps the unit fraction above it. This is useful when comparing the two unit fractions.

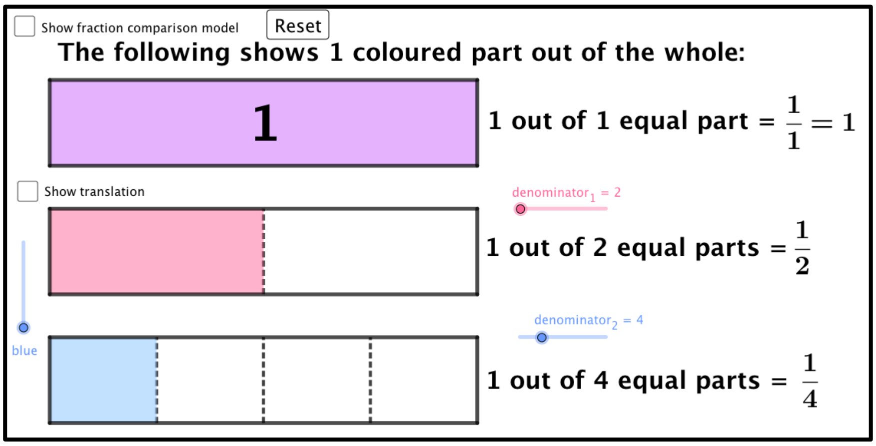

FIGURE 1: Bar model applet – basic setting

When the 'Show fraction comparison model' box in the top left corner is ticked one enters a more complex/flexible setting of the applet. In this 'fraction comparison model' two rectangular bars are shown, and the numerator and denominator of each can be changed using the sliders provided (Figure 2).

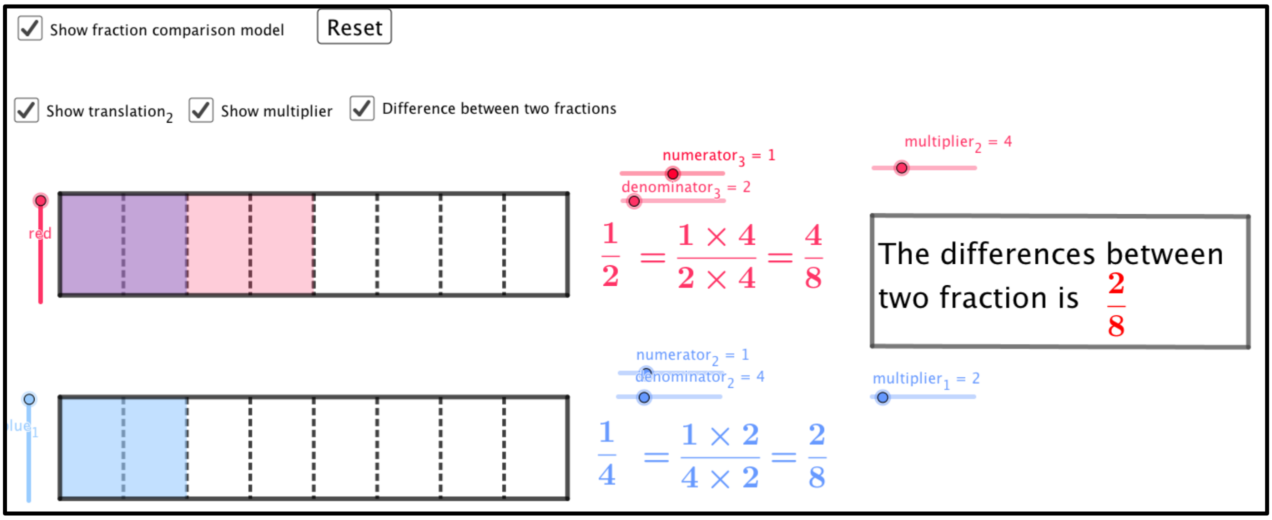

FIGURE 2: Bar model applet – fraction comparison model

A 'show translation' check box as well as two sliders on the left are provided to allow either shaded fraction to be superimposed on the other. A 'show multiplier' check box is also provided. When this box is ticked one can express each of the two fractions in different yet equivalent forms by scaling the numerator and denominator by the same constant. An extra check box labelled 'Difference between two fractions' is also provided. The bar model applet can be accessed and explored using the following link: https://www.geogebra.org/m/cgsraqzm

CIRCULAR MODEL

This applet makes use of circles rather than bars to illustrate fractions. In the basic setting of the circular model applet (Figure 3), two circles are shown. The top circle is fixed and is provided to illustrate one whole. For the second circle, sliders are provided so that both the numerator and the denominator can be changed. The numerator is restricted so that it can only take on values less than or equal to the denominator.

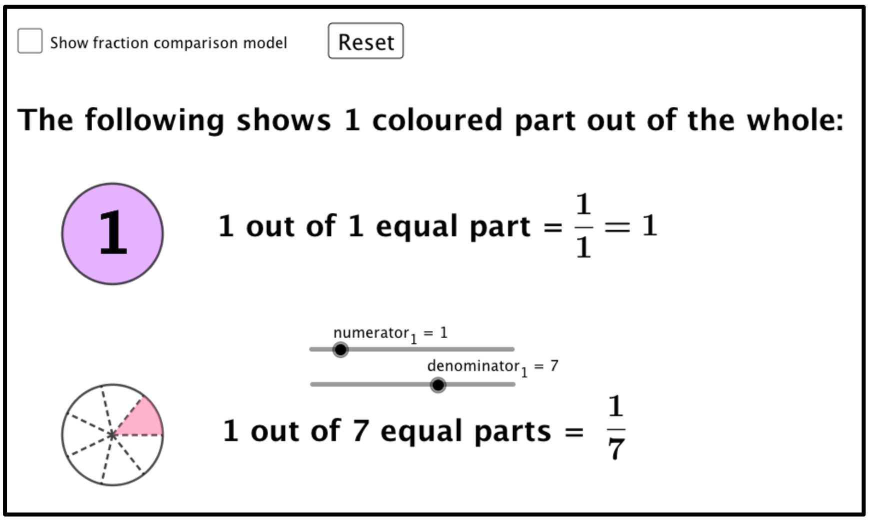

FIGURE 3: Circular model applet – basic setting

When the 'Show fraction comparison model' box in the top left corner is ticked one enters a more complex/flexible setting of the applet. In this 'fraction comparison model' two circles are shown, and the numerator and denominator of each can be changed using the sliders provided (Figure 4). In this setting of the applet one can also create improper fractions by setting the numerator to be greater than the denominator. When the 'Compare' box is ticked, the fraction on the left can be dragged across to the right in order to superimpose the two fractions. This is useful when visually comparing two fractions.

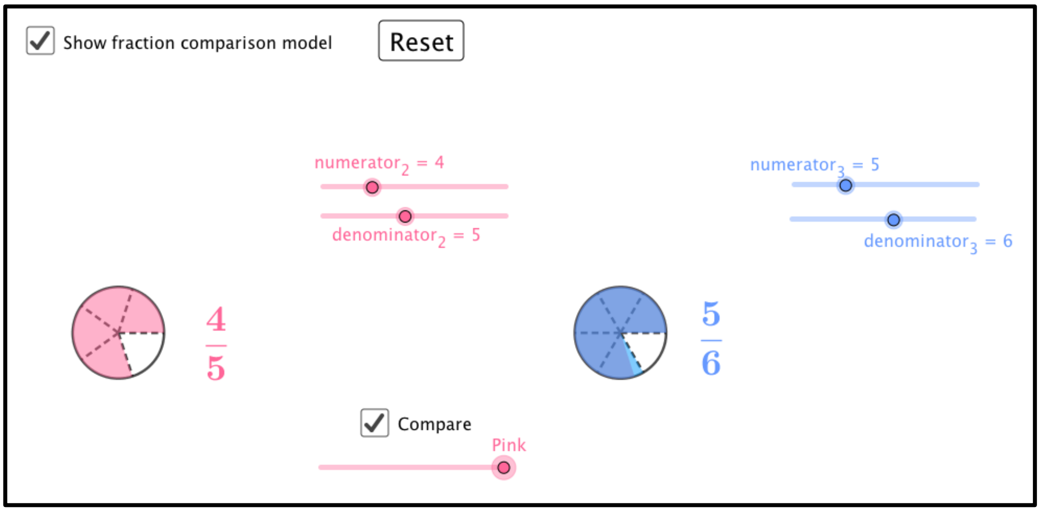

FIGURE 4: Circular model applet – fraction comparison model

The circular model applet can be accessed and explored using the following link:
https://www.geogebra.org/m/hwfkq9bb

USING THE APPLETS IN THE CLASSROOM

The different applets can be used flexibly, based on the needs of different students in the class. The applets can be used as an introduction to fractions – where pupils can individually explore the visual effect of changing numerators and denominators – or they can be used to support/initiate the following observations:

- When the numerators of the fractions are the same, the larger the denominator, the smaller the value of the fraction.
- When the denominators of the fractions are the same, the larger the numerator, the larger the value of the fraction.
- When the numerator is greater than the denominator, the fraction is greater than 1.
- When multiplying both the numerator and the denominator by the same number, the value of the fraction is not changed.

A useful approach when students engage dynamically with the fraction models is to compare fractions where the numerator is the same, and to compare fractions where the denominator is the same. One can then build on this by asking more complex questions, such as deciding which is the larger fraction, $\frac{4}{6}$ or $\frac{5}{7}$. One could investigate this initially by overlaying the two fractions using the fraction comparison model (Figure 5).

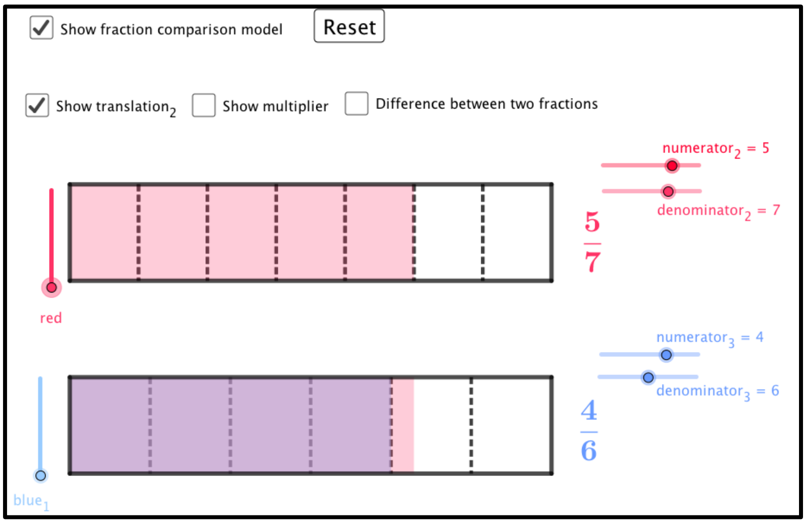

FIGURE 5: Comparing $\frac{5}{7}$ and $\frac{4}{6}$ using the bar model applet.

Having established visually that $\frac{5}{7}$ is bigger than $\frac{4}{6}$ one can turn the discussion to the non-shaded regions, as opposed to the shaded regions which represents the fractions. The non-shaded regions are $\frac{2}{7}$ and $\frac{2}{6}$ respectively. Since these two fractions have the same numerator we can compare them directly. Since $\frac{2}{7}$ is smaller than $\frac{2}{6}$ (since it has a larger denominator) it follows that that $\frac{5}{7}$ is larger than $\frac{4}{6}$. This sort of reasoning can be supported by the dynamic exploration of fractions using these applets.

CONCLUDING COMMENTS

The purpose of this article was to share and describe a number of specifically designed GeoGebra applets that allow students to engage dynamically with fractions. We have found these applets particularly useful as they add an important dynamic element to the pictorial visualization of fractions.

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Mathematics Assessment during the COVID-19 Pandemic

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One of the many consequences of the COVID-19 pandemic in South Africa, as was the case in most affected countries around the world, was that schools were forced to close to help curb the rate of infection. With teachers and students forced to stay at home, mathematics teachers were obliged to adapt their teaching and assessment practices, something that mathematics education has been advocating for decades. It did not take long before discussions around how teachers could adapt their assessment practices gained traction in the mathematics community. From my interaction with colleagues I came to realise that the predominant challenges faced by many mathematics teachers were not only technological but also of a pedagogical nature. Let us unpack this together.

Mathematics teachers at some schools have been fortunate enough to be able to provide learners with virtual lessons and online learning resources. Many teachers developed video lessons and online resources for students, and virtual lessons became popular to assist the students with online-learning[2]. However, it did not take long before questions were raised by teachers regarding online assessment. Sentiments such as "I can get my head around teaching online but not online assessment" gathered momentum in the mathematics community, and such concerns became a popular thread in a mathematics teaching e-mail group. From my observations, mathematics teachers were able to adjust and adapt their instructional strategies far more readily than their assessment practices. Indeed, many teachers were somewhat sceptical of the role of assessment during this online learning period.

There is the danger of us weakening the relationship between teaching, learning and assessment because our traditional measuring instruments are no longer useable and compatible with the learning that is taking place. Focusing on mathematics assessment is essential because what mathematics teachers assess relates to what they value as teachers. As has been argued, assessment not only places value on things but emphasizes what we value. In this respect it is critical to ask if we indeed measure what we value, or whether we are simply measuring what we can measure easily and thus end up valuing what we can measure. There have been increasing calls, both in the classroom assessment literature as well as the mathematics education literature, for teachers to make changes to their assessment practices in relation to promoting a learning culture.

Questions from mathematics teachers asking for assistance on e-mail groups around the validity, reliability and trustworthiness of assessing students during this remote learning period, mostly via online modes, became louder and even more desperate. Many teachers expressed the pressure they experience from school management to continue assessing as 'normal'. Assessing as 'normal' is, of course, synonymous with using summative assessment instruments. While I am certainly not discrediting summative assessments, I nonetheless want to highlight that focusing on the *purpose* of assessment is vital to eradicate conflict between formative classroom-based assessments and summative assessments.

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[2] While I acknowledge that there are distinctions between "e-learning", "remote learning" and "distance learning", in this article I use these terms interchangeably, and under the same umbrella.

There should be a clear distinction between using "assessment *for* learning", "assessment *as* learning", and "assessment *of* learning". Our assessments must fulfil a greater purpose than merely collecting data in the form of marks for reporting and certification purposes. Although a classroom-based assessment may be designed and packaged as formative (assessment for learning) or summative (assessment of learning), it is the actual methodology, data analysis, and use of the results that really determine whether an assessment is formative or summative.

It interested me that the challenges faced by many mathematics teachers were not simply of a technological nature but also of a pedagogical nature pertaining to their conceptions of assessment. Teachers' conceptions[3] of assessment are formed by an entangled network of beliefs, including beliefs of teaching and learning mathematics, beliefs of the purposes of assessment, and beliefs of the expectations and ability of their students. I argue that mathematics teachers' conceptions of assessment will determine the extent to which they are likely to adapt or incorporate new ideas into their assessment practices. Many studies have found that mathematics teachers' conceptions of assessment are the strongest indication of whether a teacher is aligned with an *assessment culture* or a *testing culture of assessment*. A significant number of aspects shape teachers' conceptions of assessment, which results in teachers having either societal conceptions of assessment or pedagogical conceptions of assessment.

Teachers will be able to adapt and incorporate new ideas into their assessment practices during this online learning phase only if they have pedagogical conceptions of assessment. Pedagogical conceptions of assessment allow for assessment to be used as a tool for improving learning, and to advocate a learning-centred culture, whereas teachers' societal conceptions will continue to create conflict when using a "testing culture" of assessment for learning purposes. An assessment culture of assessment (as opposed to a testing culture of assessment) is critical during this online learning phase in order to integrate learning and teaching. The assessments designed and used by teachers must foresee the specific type of feedback and assessment data that the assessment will generate, to provide meaningful feedback about each student's progress. For teachers to have pedagogical conceptions of assessment, which aligns to an assessment culture, qualitative and descriptive assessment feedback (as opposed to mere quantitative feedback) is critically important.

Not all teachers are of course in a position to assess their students via online means. For teachers who are in such a position, technology is of course an obstacle but it should not be the biggest obstacle. There is no reason why teachers should not be able to assess their students in a manner that will promote learning if they have an assessment culture of assessment as opposed to a testing culture of assessment. There is a range of free and easily accessible resources available to support teachers to design learning and assessments via online platforms, including Google forms and Microsoft Quiz. What becomes important is to use these online platforms in a way that transcends mere emulation of traditional forms of assessment, and where the focus becomes the quality of the feedback that will be generated in order to bring teaching and learning closer together.

It must be noted that although teachers are viewed as essential agents of change in the ongoing attempt to reform education, they are also significant obstacles to change and reforming education. One of the very few positive outcomes of the COVID-19 pandemic for mathematics education was that teachers had to evaluate and adapt their assessment practices. I urge every teacher who is in a position to assess their students via online means to continuously evaluate their conceptions of assessment. The symbiotic relationship between assessment in mathematics and the learning of mathematics is of the essence during these unprecedented times, and it will require a shift in our pedagogy.

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[3] I use the term conception to describe general mental structures, encompassing beliefs, meanings, concepts, propositions, rules, mental images and preferences. Beliefs, therefore, represent a sub-category of conceptions.

Rotating a Square and Rectangle

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Imagine a unit square positioned in the first quadrant of the Cartesian plane as illustrated in Figure 1. The four vertices of the square are the points $(0;0)$, $(0;1)$, $(1;1)$ and $(1;0)$.

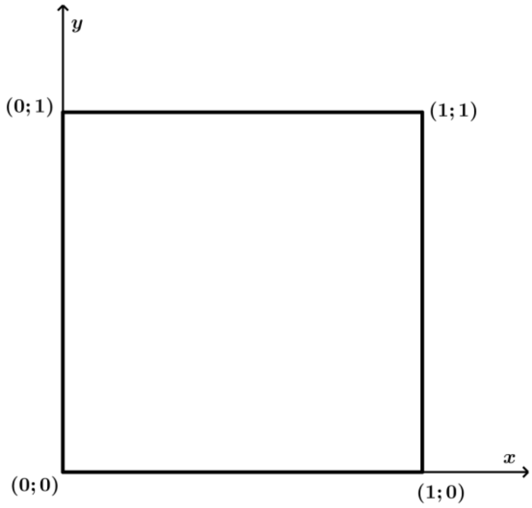

FIGURE 1: A unit square positioned in the first quadrant

Now imagine rotating the square in a clockwise direction such that one of the vertices is always in contact with the x-axis, and another is always in contact with the y-axis. What curve would be traced out by the top left vertex, originally at $(0;1)$, as the square rotates through a quarter turn? Figure 2 illustrates this process and traces the locus of the vertex in question.

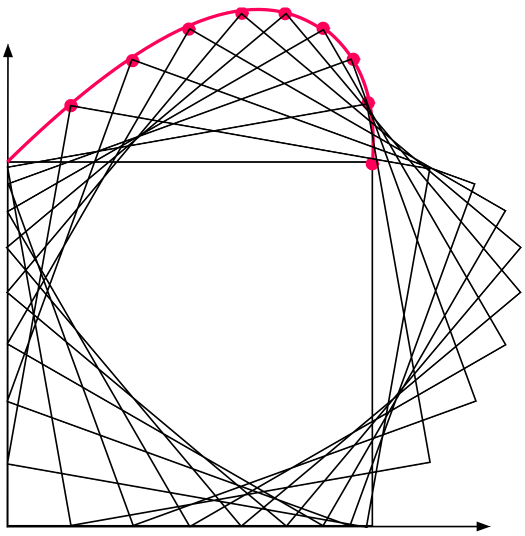

FIGURE 2: Rotating a square through a quarter turn

Would it be possible to determine an explicit formula for the path illustrated in Figure 2? Let's consider a general unit square in the process of rotating, as illustrated in Figure 3. The vertex tracing out the path we are interested in is $A(x; y)$. The vertex in contact with the x-axis is $E(p; 0)$, and the vertex in contact with the y-axis is $C(0; q)$.

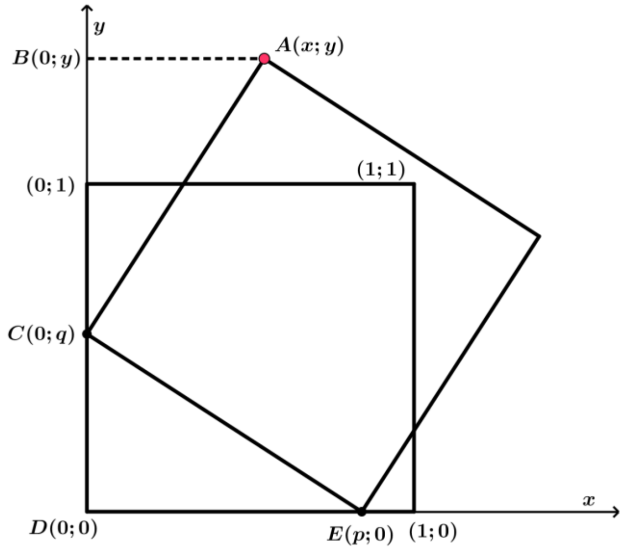

FIGURE 3: Generalising the rotating square

Since the side length of the square is 1 unit, it is clear that $p^2 + q^2 = 1$. From this we can express p in terms of q as $p = \sqrt{1 - q^2}$. Since triangles ABC and CDE are similar (in fact they are congruent), it follows that:

$$\frac{y - q}{p} = \frac{x}{q} = 1$$

From this we can write $y = \left(\frac{p}{q}\right)x + q$ and note that $q = x$. Substituting $p = \sqrt{1 - q^2}$ into $y = \left(\frac{p}{q}\right)x + q$ gives $y = \left(\frac{\sqrt{1 - q^2}}{q}\right)x + q$. Finally, since $q = x$, we can write $y = x + \sqrt{1 - x^2}$. The graph of this function is illustrated in Figure 4.

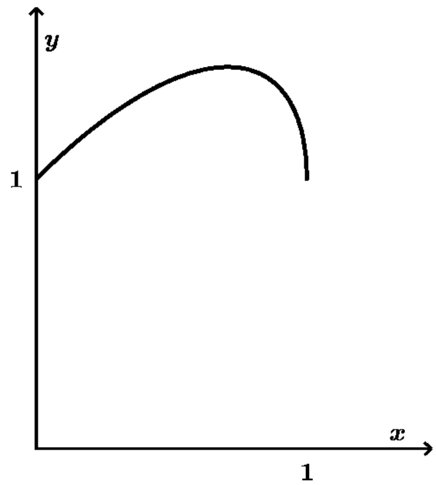

FIGURE 4: The graph of $y = x + \sqrt{1 - x^2}$ for $x \geq 0$

Let us now extend this idea to a more general rectangle. Consider a rectangle with length 1 and width w. As in the case of the square, what path would be traced out by the top left vertex as the rectangle rotates through a quarter turn? Figure 5 illustrates the locus of the vertex in question.

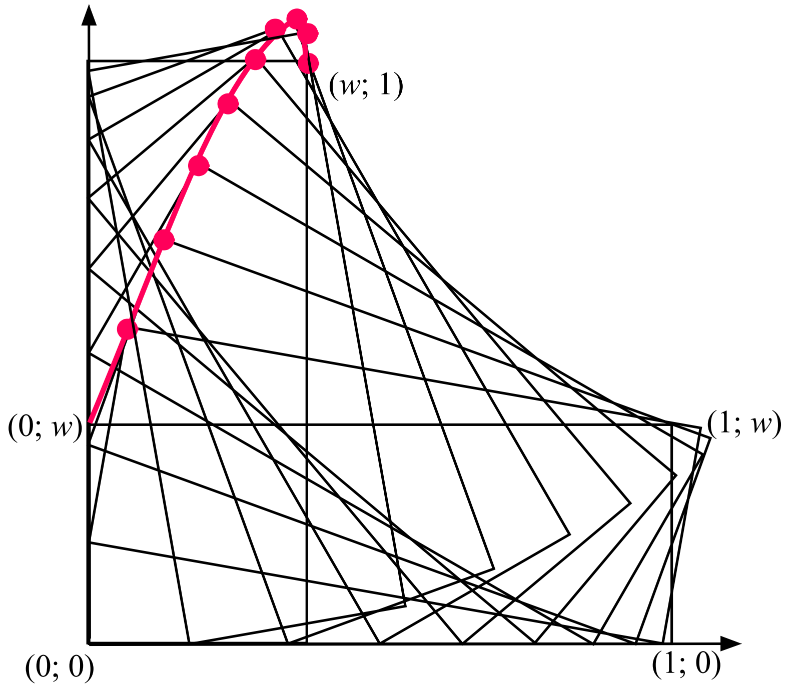

FIGURE 5: Rotating a rectangle through a quarter turn

As with the square, let us try to establish an explicit formula for the path illustrated in Figure 5. With reference to Figure 6, the vertex tracing out the path we are interested in is $A(x; y)$. The vertex in contact with the x-axis is $E(p; 0)$, and the vertex in contact with the y-axis is $C(0; q)$.

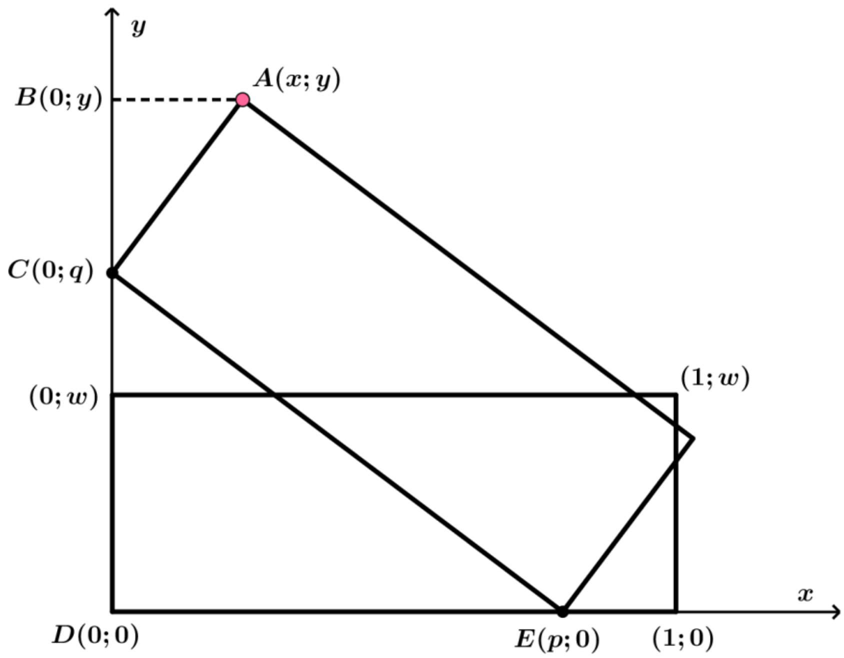

FIGURE 6: Generalising the rotating rectangle

Since the length of the rectangle is 1 unit, it is clear that $p^2 + q^2 = 1$. From this we can express p in terms of q as $p = \sqrt{1 - q^2}$. Since triangles ABC and CDE are similar, it follows that:

$$\frac{y - q}{p} = \frac{x}{q} = \frac{w}{1}$$

From this we can write $y = \left(\frac{p}{q}\right)x + q$ and see that $q = \frac{x}{w}$. Substituting $p = \sqrt{1 - q^2}$ into $y = \left(\frac{p}{q}\right)x + q$ gives $y = \left(\frac{\sqrt{1 - q^2}}{q}\right)x + q$. Finally, since $q = \frac{x}{w}$, we can write:

$$y = \left(\frac{\sqrt{1 - \left(\frac{x}{w}\right)^2}}{\frac{x}{w}}\right)x + \frac{x}{w}$$

This we can simplify to $y = \frac{x}{w} + w\sqrt{1 - \left(\frac{x}{w}\right)^2}$. Figure 7 illustrates graphs of this function for different values of w, for $x \geq 0$.

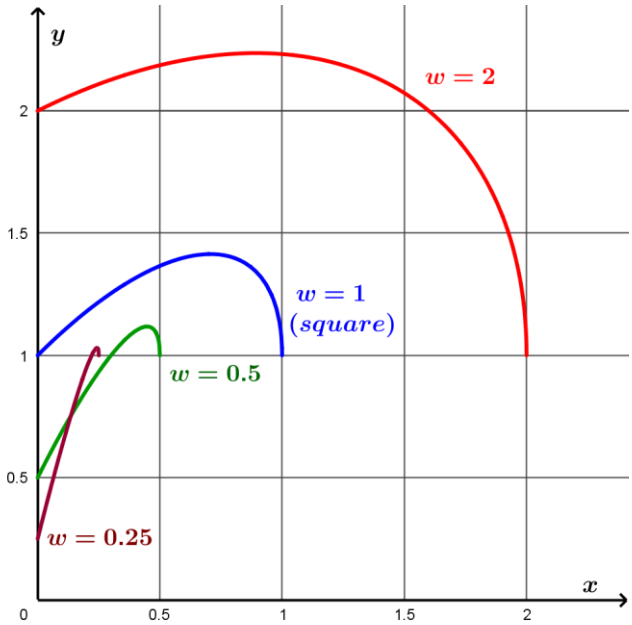

FIGURE 7: Graphs for rectangles with different widths

Note that as w gets smaller the greater the initial gradient of the slope. In each case the function increases to some maximum value and then ends with a "hook". One could readily use calculus to determine the point at which the function is a maximum, but our intuition suggests that the maximum is reached when the diagonal of the rectangle is vertical, meaning the maximum height is $\sqrt{w^2 + 1}$. It is left for the reader to verify this.

ACKNOWLEDGEMENT

My thanks to Samuel G. Camp III who suggested the problem to me. He also assisted me in the visualization of the shape of the curves by roughly plotting the points using graph paper and rectangular cut-outs.

Using Logarithms to Find the Length of Large Numbers

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INTRODUCTION

Some large numbers, such as *millions* and *billions*, have real references in human experience and are encountered in a number of different contexts. Bigger numbers, as they move outside of the realm of human experience, become more difficult to conceptualise. Nonetheless, large numbers abound in our world and are of great mathematical interest and importance. A useful means of working with large numbers is by employing logarithms – the word itself deriving from the Greek *logos* (ratio) and *arithmos* (number). The method of logarithms was first publicly set forth by the Scottish mathematician John Napier in 1614, and was rapidly adopted by navigators, scientists, engineers, and many others as a means of more easily carrying out tedious calculations. While the basic log laws should be familiar to Grade 12 students, it is the purpose of this article to showcase the usefulness of logarithms in the context of large numbers.

QUESTION 1

How many digits does 1234^{5678} have when it is written out in full as a decimal number?

THEOREM 1

In order to answer the above question we need to make use of the following theorem:

> Suppose a and b are positive integers. Then the length of a^b is 1 plus the integer part of $\log a^b$.

This theorem is based on the fact that if m is an integer such that $10^n \le m < 10^{n+1}$, where n is a non-negative integer, then the integer m has $n+1$ digits when written as a decimal number. Using this fact we can prove the above theorem as follows. Assume that $a^b = 10^r$ for some real number r. Thus $r = \log a^b$. If the integer part of r is n, then $n \le r < n+1$. From this we can write $10^n \le 10^r < 10^{n+1}$. We thus have $10^n \le a^b < 10^{n+1}$, from which it follows that a^b has $n+1$ digits when written out in full as a decimal number.

SOLUTION 1

Let us now return to our question, namely how many digits 1234^{5678} has when written out in full as a decimal number. Using the power law we first calculate $\log 1234^{5678}$ as follows:

$$\log 1234^{5678} = 5678 \times \log 1234 = 17\,552{,}487\ldots$$

The integer part of this answer is $17\,552$, which means that 1234^{5678} has $17\,552 + 1 = 17\,553$ digits when written out in full as a decimal number.

QUESTION 2

How many bits does the number 1234^{5678} have when it is written out in full in binary form? The length of binary numbers is important to know as this tells us how many bits of memory would be needed to store the number electronically (there are 8 bits to 1 byte).

THEOREM 2

In order to answer the above question we need to make use of the following theorem, which is a more general form of Theorem 1 and relies on the fact that if m is an integer such that $c^n \leq m < c^{n+1}$, where n is a non-negative integer, then the integer m has $n+1$ digits when written out in full as a base c number.

> Suppose a and b are positive integers. Then the length of a^b in base c is 1 plus the integer part of $\log_c a^b$.

SOLUTION 2

Let us now return to our question, namely the number of bits needed to store the number 1234^{5678} when it is written in binary form. Using the power law we first calculate $\log_2 1234^{5678}$ as follows:

$$\log_2 1234^{5678} = 5678 \times \log_2 1234 = 58\,308{,}101\,\ldots$$

The integer part of this answer is $58\,308$, which means that 1234^{5678} requires $58\,308 + 1 = 58\,309$ bits of data to be stored electronically.

QUESTION 3

What is the first digit of the number 1234^{5678} when it is written out in full as a decimal number?

THEOREM 3

In order to answer the above question we need to make use of the following theorem:

> Suppose a and b are positive integers. Let f be the fractional part of $\log a^b$. The first digit of a^b is the integer part of 10^f.

This theorem is based on the fact that if a^b has $n+1$ digits, then the integer part of $\frac{a^b}{10^n}$ is the first digit of a^b. Using this fact we can prove the theorem as follows. Assume that $a^b = 10^r$ for some real number r. We can thus write $r = \log a^b$. If the integer part of r is n, then $n \leq r < n+1$. From this we can write $10^n \leq 10^r < 10^{n+1}$. Thus $10^n \leq a^b < 10^{n+1}$, from which it follows that a^b has $n+1$ digits when written out in full as a decimal number. Letting f be the fractional part of $\log a^b$, then $f = r - n$. But since we have $a^b = 10^r$ for some real number r, we can write $\frac{a^b}{10^n} = \frac{10^r}{10^n} = 10^{r-n} = 10^f$. Thus the first digit of a^b is the integer part of 10^f. It is worthwhile noting that this theorem can be modified to determine not only the first digit of a large number expressed in exponent form, but the first few digits of such a number. It is also worth noting that the theorem can be generalised to other bases – the first digit of a^b written out in full in base c is the integer part of c^f where f is the fractional part of $\log_c a^b$.

SOLUTION 3

Let us now return to our question, namely determining the first digit of the number 1234^{5678} when it is written out in full as a decimal number. Making use of the above theorem, we are interested in the fractional part of $\log 1234^{5678} \approx 17\,552{,}487$, i.e. $0{,}487$. We now calculate $10^{0{,}487} \approx 3{,}072$ correct to three decimal places. The required first digit of the number 1234^{5678} is thus 3.

Deriving the Quadratic Formula: A Visual Approach

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INTRODUCTION

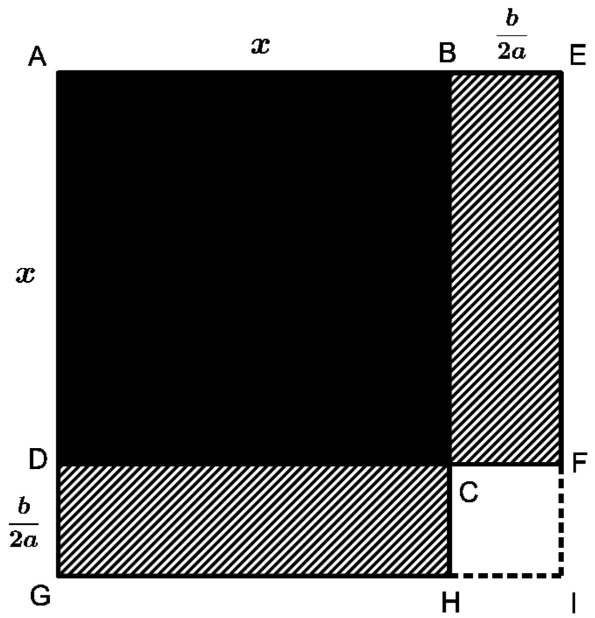
In this article I present a simple visual approach to deriving the quadratic formula, i.e. the general expression for the roots of the quadratic equation $ax^2 + bx + c = 0$ where a, b and c are constants and $a \neq 0$. This visual approach provides an alternative to the purely algebraic derivation of the formula most commonly described in secondary school mathematics textbooks.

THE VISUAL APPROACH

Since $a \neq 0$, we can divide through by a and rewrite the equation $ax^2 + bx + c = 0$ in the form:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

The expression on the left-hand side of the equation can be represented by the total shaded region in the following diagram, i.e. the sum of the area of square $ABCD$ and the area of rectangles $BEFC$ and $DCHG$.



$$Total\ shaded\ area = x^2 + 2\left(x.\frac{b}{2a}\right) = x^2 + \frac{b}{a}x$$

Since the total shaded area can also be expressed as the area of square $AEIG$ minus the area of square $CFIH$, we can also write:

$$Total\ shaded\ area = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

We thus have:

$$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2=x^2+\frac{b}{a}x$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2=-\frac{c}{a}$$

Restructuring this we have:

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore\ \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

By square rooting both sides and making x the subject, we arrive at the familiar quadratic formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Concluding comments

This diagrammatic approach, while similar to the visual analogue of 'completing the square' (Samson, 2013), provides an alternative to the purely algebraic derivation of the quadratic formula as it appears in most secondary school mathematics textbooks.

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The Value of using Signed Quantities in Geometry

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INTRODUCTION

In the previous issue of *Learning and Teaching Mathematics,* it was shown in De Villiers (2020) how a formula for the area of a quadrilateral in terms of its diagonals and the sine of the angle between them can easily be extended to concave or crossed quadrilaterals with the use of 'signed' or 'directed' areas.

While the idea of negative quantities is familiar, and taken for granted in arithmetic, algebra, trigonometry, and even in physics, the mere mention of concepts in geometry like 'negative' areas, distances or angles often generates incredulous looks among people who hear of it for the first time. Of course, the ancient Greeks never considered the possibility of negative quantities in geometry, but at least since about the 19th century, geometers have gradually grown accustomed to this 'revolutionary' idea, mainly because of its value in providing one general proof, rather than having to consider and prove several different cases.

The purpose of this article is to give two further examples of the value of directed or signed quantities in geometry, focusing this time on directed angles and directed distances.

FIRST EXAMPLE: DIRECTED ANGLES

The first example is a well-known theorem prescribed in most school mathematics curricula around the world, including the South African High School Mathematics Curriculum, where it is usually dealt with in Grade 11/12.

Theorem: The angle subtended by an arc (or chord) at the centre of a circle is twice the size of the angle subtended by the same arc (or chord) at the circumference of the circle (on the same side of the chord as the centre).

Proof: For a complete proof of the result, one has to consider three different geometric configurations as shown in Figure 1.

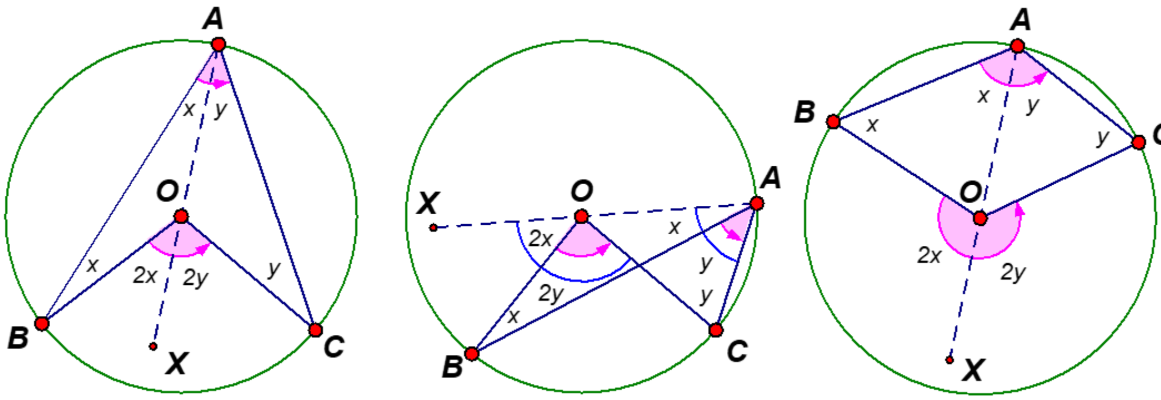

FIGURE 1: Three different cases for the 'Angle at Centre' theorem

When proving the result, teachers and textbooks sometimes only consider the first case in Figure 1, but this is not sufficient. The proof of the third case is identical to the first case, although students often have difficulty in visualizing and correctly applying the theorem when the central angle becomes reflexive. However, the proof of the second case is quite different from the other two.

The proofs for the first and third cases in Figure 1 are the same. For example, draw AO and extend to X as indicated. Then:

$$\angle OBA = x = \angle OAB \ldots OB = OA;\ \text{radii}$$

$$\Rightarrow\ \angle BOX = 2x \ldots \text{exterior angle of } \Delta ABO$$

$$\angle OCA = y = \angle OAC \ldots OC = OA;\ \text{radii}$$

$$\Rightarrow\ \angle XOC = 2y \ldots \text{exterior angle of } \Delta AOC$$

$$\text{Thus, } \angle BOC = \angle BOX + \angle XOC = 2x + 2y = 2(x + y) = 2\angle BAC$$

However, the second case is different, and involves the subtraction of angles. Although the first part of the proof is identical, the difference is apparent in the last step:

$$\angle OBA = x = \angle OAB \ldots OB = OA;\ \text{radii}$$

$$\Rightarrow\ \angle BOX = 2x \ldots \text{exterior angle of } \Delta ABO$$

$$\angle OCA = y = \angle OAC \ldots OC = OA;\ \text{radii}$$

$$\Rightarrow\ \angle XOC = 2y \ldots \text{exterior angle of } \Delta AOC$$

$$\text{Thus, } \angle BOC = \angle XOC - \angle BOX = 2y - 2x = 2(y - x) = 2\angle BAC$$

Using the idea of 'directed' (or 'signed') angles, however, one does not need to write down the second proof, and the first proof suffices for all three cases, provided one clearly states at the beginning that one is making use of directed angles.

The idea is really quite simple. Note that in the first and third cases the size of $\angle BOX$ is determined by an *anti-clockwise* rotation of ray *OB* around *O* to map onto ray *OX* (which in trigonometry is normally defined as a positive rotation). However, in the second case the size of $\angle BOX$ is determined by a *clockwise* rotation of ray *OB* around *O* to map onto ray *OX* (which in trigonometry is normally defined as a negative rotation). In other words, in the second case, $\angle BOX$ can be viewed as negative in relation to $\angle BOX$ in the other two cases. Most significantly, the first proof therefore holds provided we regard $\angle BOX$ as negative in the second case, as the first proof then automatically covers the necessary subtraction in the second case.

One of the advantages therefore of using directed angles is that it avoids having to write down several different proofs in order to cover different cases. It is important, however, that one clearly states at the outset of such a proof that one is assuming directed angles.

Most dynamic geometry packages allow for the measurement of directed angles. Although the default setting of angle measurement in *Sketchpad* is for the absolute value of an angle, this can easily be changed in the Edit/Preferences/Units/Angle menu to 'directed degrees'. The popular GeoGebra actually has 'directed degrees' as its default measurement. For example, when a quadrilateral *ABCD* is changed into a crossed quadrilateral (as shown in Figure 2) by dragging vertex *C* across *AD*, two reflex angles are formed at the vertices *C* and *D*, and the angle sum of the angles of the crossed quadrilateral becomes 720°.

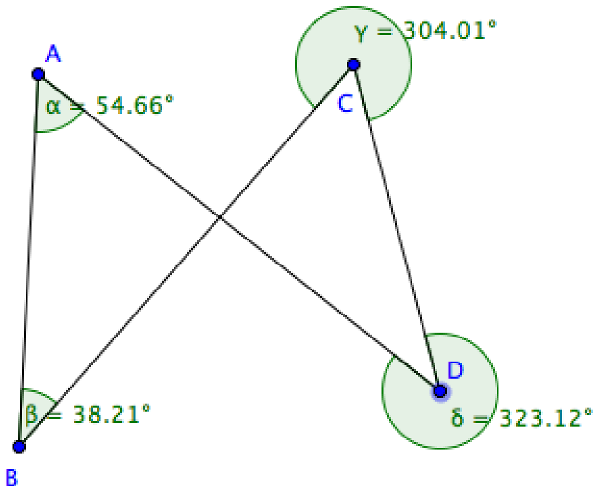

FIGURE 2: The angles of a crossed quadrilateral in GeoGebra

Instead of the usual practice of 'monster-barring' crossed quadrilaterals, such a surprising empirical discovery about crossed quadrilaterals by learners could create an excellent opportunity for not only learning about directed angles, but also about explaining why (proving that) the angle sum is 720° (De Villiers, 2003, pp. 40-44; 156-157).

SECOND EXAMPLE: DIRECTED DISTANCES

Consider the following problem from De Villiers (2003, pp. 26, 149-150), which is easily accessible for learners in Grades 8-9: Prove that the sum of the distances[4] from a point to the sides of a parallelogram is constant. A dynamic geometry sketch is available online for the reader or learners to explore:

http://dynamicmathematicslearning.com/parallelogram-distances.html

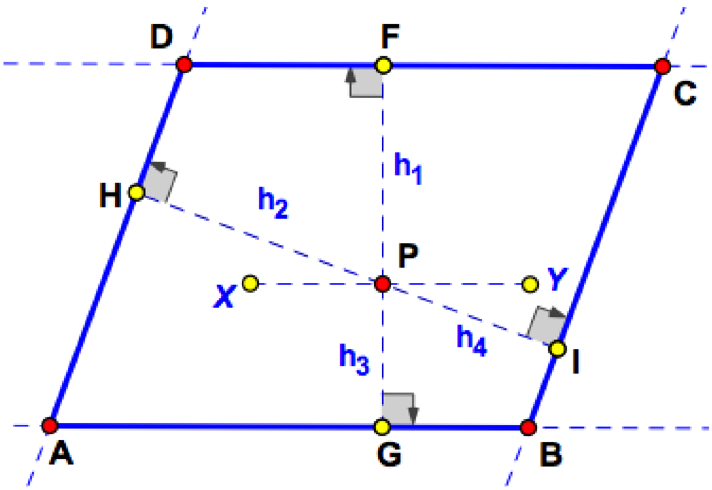

FIGURE 3: Distances to the sides of a parallelogram.

[4] With 'distances' here is meant 'shortest distances', which are the perpendiculars from P to the sides.

Proof: Consider Figure 3 showing parallelogram *ABCD* with an arbitrary point *P*, and the distances from *P* to the sides. Although in the accurately drawn diagram in Figure 3 it is clear that *FPG* and *HPI* are straight lines, we may not assume they are straight – we need to prove[5] it. Draw line *XPY* parallel to the opposite sides *AB* and *CD*. Then it follows that:

$$\angle XPF = 90° = \angle DFP \text{ (co-interior } \angle\text{s)}$$

$$\angle XPG = 90° = \angle BGP \text{ (alternate } \angle\text{s)}$$

$$\Rightarrow \angle XPF + \angle XPG = 90° + 90° = 180°$$

Hence, *FPG* is a straight line. In the same way, it can be shown that *HPI* is also straight. Now note that:

$$h_1 + h_3 = \text{constant} = c_1 \ldots \text{ distance between parallels } AB \text{ and } CD \text{ is constant}$$

$$h_2 + h_4 = \text{constant} = c_2 \ldots \text{ distance between parallels } AD \text{ and } BC \text{ is constant}$$

$$\Rightarrow h_1 + h_2 + h_3 + h_4 = c_1 + c_2 = \text{constant.}$$

This completes the proof.

What happens when point *P* is dragged outside the parallelogram *ABCD*? The reader is requested to explore this now in the dynamic sketch at the URL provided earlier.

While the sum of the distances remains constant as long as *P* is inside *ABCD*, the reader will find as soon as *P* is moved outside *ABCD,* as shown in Figure 4, that this sum is no longer constant. So is the theorem only valid while *P* is inside the parallelogram?

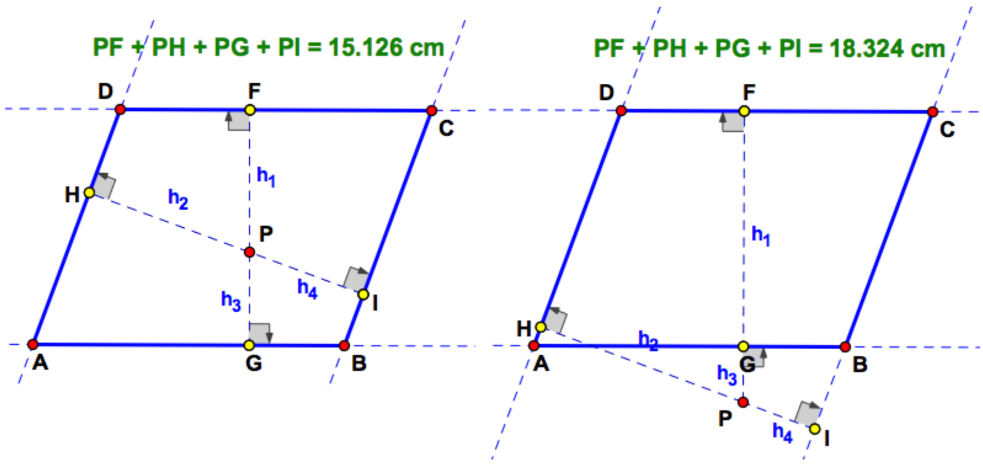

FIGURE 4: Moving point *P* outside the parallelogram

If we use *directed distances* (or *vectors*) it is easy to see that the result actually still holds when *P* is moved outside *ABCD*, because it results in a change of direction for some of the distances. Consider Figure 5, which shows vectors $\overrightarrow{GP}$ and $\overrightarrow{PF}$ with the resultant of $\overrightarrow{GP} + \overrightarrow{PF} = \overrightarrow{GF}$. Note that in the second case, when *P* is moved outside the parallelogram, $\overrightarrow{GP}$ has changed direction and the magnitude of $\overrightarrow{GP}$ now needs to be subtracted from the magnitude of $\overrightarrow{PF}$. Hence, the resultant $\overrightarrow{GF}$ remains unchanged[6].

[5] Pedagogically, when proving it in class, it is probably advisable rather to make use of an inaccurately drawn sketch so that these lines do NOT appear straight.

[6] Sometimes instead of vectors, the convenient definition is used that all distances falling completely outside a figure are regarded as negative. For example, note that *PG* changes direction as soon as *P* is moved outside the parallel lines, and can therefore be considered as negative.

Since the same argument applies for the sum of the directed distances (vectors) $\overrightarrow{IP}$ and $\overrightarrow{PH}$, the total sum of the directed distances (vectors) from P to the sides remains constant, even when P is moved outside *ABCD*.

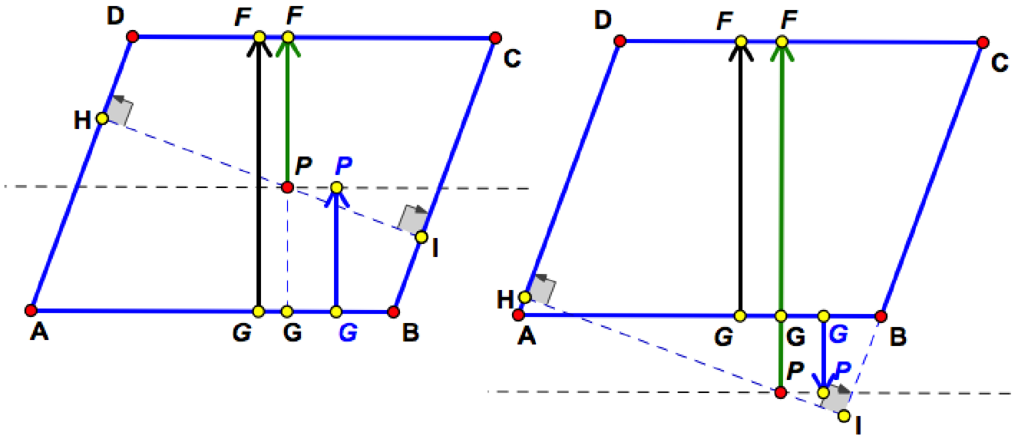

FIGURE 5: Using directed distances/vectors

As we have seen in this example, the value of using directed distances (or vectors) in geometry is that many results can be extended so that they hold more generally. Viviani's theorem, for example, that states that the sum of the distances from a point to the sides of an equilateral triangle is constant (see Samson, 2012), can similarly be generalized by using directed distances so that it is also valid when the point is moved outside the triangle. This also applies to generalizations of Viviani's theorem to equi-angled and equilateral polygons, to Clough's variation of Viviani's theorem (De Villiers, 2012), as well as to generalizations of Viviani to 3D (De Villiers, 2013).

Finally, since this result for a parallelogram follows from the 'constant distance' property of pairs of parallel lines, it is easy to see that it generalizes to any $2n$-gon with opposite sides parallel. It therefore provides another good example of the so-called 'discovery' function of proof mentioned in De Villiers (1990), where an explanatory proof leads to further generalization.

Concluding remarks

This paper has given two examples of the value of using directed areas and distances. In the first case with directed angles, it was useful because writing down a 'directed angles' proof would automatically cover the three different configurations. In the case with directed distances, it was useful to extend the result to points outside the parallelogram. Learning about directed quantities in geometry might be particularly useful for learners who wish to participate in high-level mathematics competitions (like PAMO, SAMO and IMO). It would also be a suitable topic to address in a Mathematics Club for talented learners at a school.

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Probability – Linear Constraints and the Feasible Region

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INTRODUCTION

The article "*A Problem in Probability – Its Geometric Solution and a Surprising Result*" in LTM No. 23 (Stupel & Sigler, 2017) investigated a variation on the broken stick problem as follows:

> A stick is broken into three parts, x, y and z. What is the probability that $xy \geq z^2$?

The authors made use of an equilateral triangle together with Viviani's Theorem to geometrically prove that $P(xy \geq z^2) \approx 0{,}473$. In this article I consider the same problem, but make use of linear constraints to create a feasible region within the sample space, from which the applicable probability follows.

INVESTIGATION

Let $L = x + y + z$ represent the total length of a stick (Figure 1) from which we require $xy \geq z^2$.

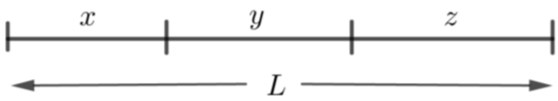

FIGURE 1: A stick of length L broken into three pieces

Since we require $xy \geq z^2$, we have:

$$xy \geq (L - x - y)^2$$

$$xy \geq L^2 - Lx - Ly - Lx + x^2 + xy - Ly + xy + y^2$$

This can be re-written in the form:

$$x^2 + y^2 - 2Lx - 2Ly + xy + L^2 \leq 0$$

The function $x^2 + y^2 - 2Lx - 2Ly + xy + L^2 = 0$ represents an ellipse with x-intercept and y-intercept both equal to L.

y-intercept $(x = 0)$: $y^2 - 2Ly + L^2 = 0 \longrightarrow (y - L)^2 = 0 \quad \therefore \quad y = L$

x-intercept $(y = 0)$: $x^2 - 2Lx + L^2 = 0 \longrightarrow (x - L)^2 = 0 \quad \therefore \quad x = L$

All co-ordinate pairs $(x; y)$ falling either on or within the circumference of this ellipse satisfy the required inequality. Figure 2 shows this ellipse with semi-major axis a and semi-minor axis b.

Since $L = x + y + z$ we have $z = L - x - y$, and since $z > 0$ we can write $0 < L - x - y$. The constraints on x and y can now be stated as follows:

$$y < -x + L \quad \ldots (1)$$

$$0 < x < L \ \ldots (2)$$

$$0 < y < L \ \ldots (3)$$

The line $y = -x + L$ together with the axis of symmetry $y = x$ are shown in Figure 2, the latter coinciding with CD, the minor axis of the ellipse. M is the point of intersection of the major axis AB and the minor axis CD. $MF = d$ is the perpendicular distance between M and the line $y = -x + L$.

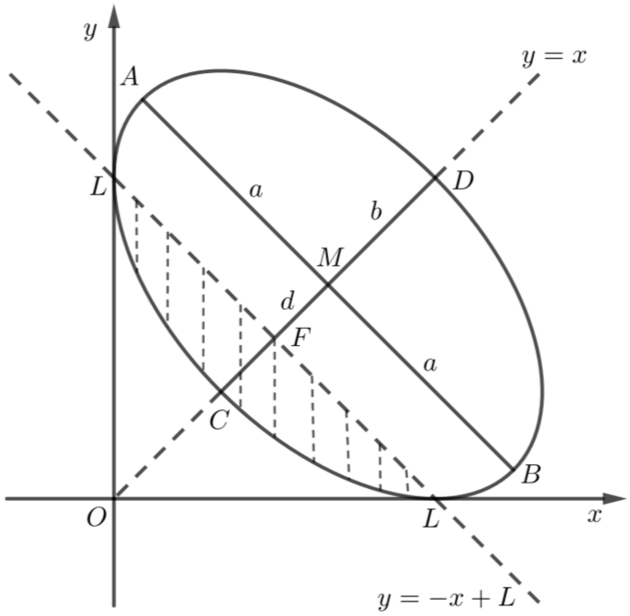

FIGURE 2: The feasible region

Co-ordinate pairs $(x; y)$ satisfying constraints (1) – (3) lie within the triangle bounded by the x-axis, the y-axis and the line $y = -x + L$, i.e. triangle LOL. However, these co-ordinate pairs also need to satisfy the ellipse inequality. The feasible region is thus the overlap of triangle LOL and the ellipse, as illustrated by the hatched region shown in Figure 2. It thus follows that:

$$P(xy \geq z^2) = \frac{Area\ of\ feasible\ region}{Area\ of\ \Delta LOL}$$

The area of triangle LOL is simple enough to calculate:

$$Area\ of\ \Delta LOL = \frac{1}{2} \times L \times L = \frac{L^2}{2}$$

Calculating the area of the hatched region is somewhat more complicated. First we need to determine the values of a, b and d in terms of L. Let us first determine the co-ordinates of points C and D, the points of intersection of the ellipse and the line $y = x$.

$$x^2 + x^2 - 2Lx - 2Lx + x^2 + L^2 = 0 \longrightarrow 3x^2 - 4Lx + L^2 = 0$$

This quadratic equation factorises to give $(3x-L)(x-L)=0$, which has solutions $x=\frac{L}{3}$ or $x=L$. We thus have the co-ordinates of points C and D as $C\left(\frac{L}{3};\frac{L}{3}\right)$ and $D(L;L)$. Since M is the midpoint of CD we thus also have $M\left(\frac{2L}{3};\frac{2L}{3}\right)$. Additionally, from symmetry, we have $F\left(\frac{L}{2};\frac{L}{2}\right)$. Using the co-ordinates of points C and D we have $CD=\frac{2\sqrt{2}L}{3}$ and thus $b=\frac{\sqrt{2}L}{3}$. Using the co-ordinates of points M and F we have $MF=\frac{\sqrt{2}L}{6}$ and thus $d=\frac{\sqrt{2}L}{6}$.

In order to determine a we first need the equation of the line AB, namely $y=-x+\frac{4L}{3}$. Substituting this into the ellipse equation gives:

$$x^2+\left(-x+\frac{4L}{3}\right)^2-2Lx-2L\left(-x+\frac{4L}{3}\right)+x\left(-x+\frac{4L}{3}\right)+L^2=0$$

This equation simplifies to $9x^2-12Lx+L^2=0$ which has solutions $x=\frac{(2+\sqrt{3})L}{3}$ or $x=\frac{(2-\sqrt{3})L}{3}$ with corresponding y-values of $y=\frac{(2-\sqrt{3})L}{3}$ and $y=\frac{(2+\sqrt{3})L}{3}$. Thus:

$$A\left(\frac{(2-\sqrt{3})L}{3};\frac{(2+\sqrt{3})L}{3}\right)\quad\text{and}\quad B\left(\frac{(2+\sqrt{3})L}{3};\frac{(2-\sqrt{3})L}{3}\right)$$

Using the co-ordinates of points A and B we have $AB=\frac{2\sqrt{6}L}{3}$ and thus $a=\frac{\sqrt{6}L}{3}$. We thus have:

$$a=\frac{\sqrt{6}L}{3}\quad;\quad b=\frac{\sqrt{2}L}{3}\quad;\quad d=\frac{\sqrt{2}L}{6}$$

It just remains to calculate the area of the hatched feasible region. In order to do this we compress the ellipse along its major axis to form a circle as shown in Figure 3. The radius of the circle is the same length of the semi-minor axis of the ellipse, and distance d remains the same.

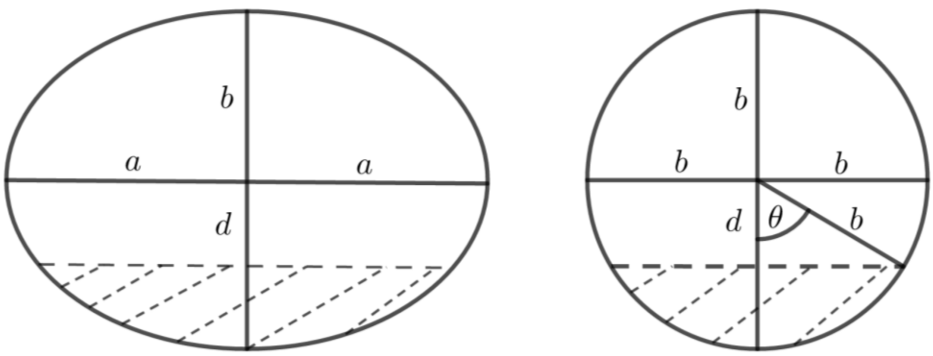

FIGURE 3: Ellipse compressed along its major axis into a circle

The following relationship applies between the ellipse and the circle:

$$Area\ of\ ellipse=\frac{a}{b}\times(Area\ of\ circle)$$

We can make use of this relationship by applying it to the hatched feasible region within the ellipse.

$$cos\theta = \frac{d}{b} = \frac{\sqrt{2}L}{6} \times \frac{3}{\sqrt{2}L} = \frac{1}{2} \quad \therefore \quad \theta = 60°$$

The area of the segment representing the feasible region in the circle is thus:

$$\frac{120°}{360°} \times \pi \times \left(\frac{\sqrt{2}L}{3}\right)^2 - \frac{1}{2} \times \frac{\sqrt{2}L}{3} \times \frac{\sqrt{2}L}{3} \times \sin 120^{\circ} = \left(\frac{4\pi - 3\sqrt{3}}{54}\right) L^2$$

We now need to scale this back up to the original ellipse. Since $a = \frac{\sqrt{6}L}{3}$ and $b = \frac{\sqrt{2}L}{3}$ we have $\frac{a}{b} = \sqrt{3}$. The area of the feasible region in the original ellipse is thus:

$$\frac{a}{b} \times \left(\frac{4\pi - 3\sqrt{3}}{54}\right) L^2 = \frac{\sqrt{3}L^2(4\pi - 3\sqrt{3})}{54}$$

Finally:

$$P(xy \geq z^2) = \frac{Area\ of\ feasible\ region}{Area\ of\ \Delta LOL} = \frac{\sqrt{3}L^2(4\pi - 3\sqrt{3})}{54} \div \frac{L^2}{2} \approx 0{,}473$$

ADDITIONAL PROBLEMS

The reader may wish to investigate the use of this method to determine each of the following probability problems:

(A) A stick is broken into three parts. What is the probability that a triangle can be constructed from the three parts? This is the well-known "broken stick problem" which has solution $P(\Delta) = 0{,}25$.

(B) A stick is broken into three parts x, y and z. What is the probability that $2xy \geq z^2$? This is similar to the question investigated in this article. However, the use of $2xy$ results in a circle inequality rather than an ellipse inequality, and as such is more aligned with core mathematics at school. The answer to the problem is $P(2xy \geq z^2) \approx 0{,}571$.

(C) Consider the set of real numbers $R \in [-2; 10]$. Numbers x and y, which may be equal, are randomly selected one after the other within R. Determine the probability that $x + y \leq \pi$. The answer to this particular problem is $P(x + y \leq \pi) \approx 0{,}177$.

CONCLUDING COMMENTS

The solution to the problem investigated in this article employed a blend of algebra and geometry. Since the probability of event A occurring in a sample space S is defined as $P(A) = \frac{n(A)}{n(S)}$ it is clear how the creation of a feasible region from a series of constraints, effectively representing $n(A)$, can assist in solving applicable problems in probability. Clearly this method can also be extended to the use of non-linear constraints.

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The Tangential or Circumscribed Quadrilateral

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INTRODUCTION

Although a substantial part of the South African school geometry curriculum focuses on exploring the properties of cyclic quadrilaterals, the study of quadrilaterals circumscribed around a circle is somewhat neglected. This is a pity, as the mathematical content falls well within reach of high school learners and utilises no geometry theorems that are not already in the curriculum. Moreover, having learners investigate such quadrilaterals provides opportunities for reinforcement of many previously learnt concepts.

A tangential or circumscribed quadrilateral can be defined simply as a quadrilateral that has an incircle (its sides therefore being tangents to the incircle). This brings us to the first easy to prove theorem involving angle bisectors.

ANGLE BISECTORS

Theorem 1: The angle bisectors of any tangential quadrilateral are concurrent at the incentre of the quadrilateral (see Figure 1).

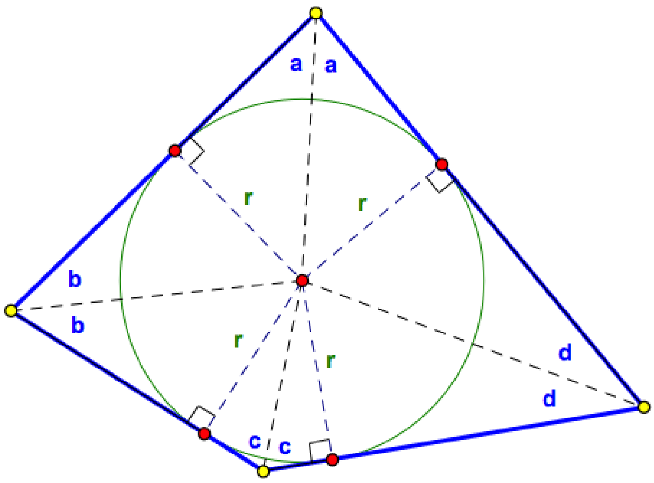

FIGURE 1: Concurrent angle bisectors

Proof:

Since the incentre is equidistant from all four sides (radii of circle are perpendicular to sides), and each angle bisector is the locus of all points equidistant from its two adjacent sides, it follows that each angle bisector must pass through the incentre.

Conversely, one should also note that concurrency of the angle bisectors is a very useful condition for a quadrilateral to be circumscribed around or tangential to a circle. For example, for a quadrilateral to have an incircle, it must have a point that is equidistant from all the sides. Therefore, the four angle bisectors must meet in a single point, i.e. be concurrent.

We actually only need to have three angle bisectors of a quadrilateral concurrent to prove that it is tangential, as the fourth angle bisector will automatically be concurrent with the other three. A teacher may give learners a ready-made geometry sketch to discover and verify this for themselves by dragging the quadrilateral until three angle bisectors are concurrent, and then clicking on a button to view the fourth. An example sketch is available online at: http://dynamicmathematicslearning.com/concurrent-angle-bisectors.html

Theorem 2: If any three angle bisectors of a quadrilateral are concurrent, then the fourth angle bisector is concurrent with them[7] (and hence, the quadrilateral is tangential).

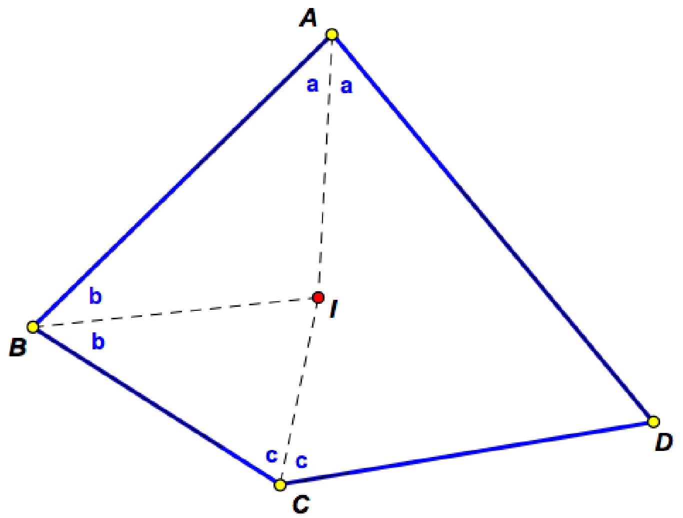

FIGURE 2: More on concurrent angle bisectors

Proof:

With reference to Figure 2, assume the angle bisectors at A, B and C of quadrilateral $ABCD$ are concurrent at I. From the properties of angle bisectors it follows that:

1) I is equidistant from sides AD and AB (lies on angle bisector of $\angle A$)
2) I is equidistant from sides AB and BC (lies on angle bisector of $\angle B$)
3) I is equidistant from sides BC and CD (lies on angle bisector of $\angle C$)

$\Rightarrow$ I is equidistant from AD and CD (transitivity)

$\Rightarrow$ I must lie on the angle bisector of $\angle D$, which completes the proof.

Note from the preceding two theorems it follows that a rhombus (and a square) are tangential since their diagonals are angle bisectors of opposite angles, and hence the incentre of the incircle would be located at the intersection of their diagonals.

Similarly, note that the axis of symmetry of a kite is an angle bisector of a pair of opposite angles. In addition, from the symmetry of the kite, it follows that the angle bisectors of the other two angles (which are reflections of each other) will intersect in a common point on the axis of symmetry. Hence, a kite also has an incircle with its incentre located on the axis of symmetry.

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[7] It is easy to generalise Theorems 1 and 2 to any n-gon where having $n-1$ angle bisectors concurrent is sufficient to prove it is tangential. In the case of the triangle, Theorem 2 proves the concurrency of its angle bisectors.

Opposite Sides

A prescribed theorem in the South African school mathematics curriculum is the following:

A (convex) quadrilateral is cyclic if and only if the opposite angles are supplementary

A different yet equivalent form of formulating this theorem is as follows:

A quadrilateral ABCD is cyclic if and only if $\angle A + \angle C = \angle B + \angle D$

This alternative formulation highlights a remarkable *angle-side* duality between the cyclic quadrilateral and the tangential quadrilateral. Whereas a cyclic quadrilateral has the two sums of opposite *angles* equal, a similar theorem in terms of the two sums of opposite *sides* holds for a tangential quadrilateral.

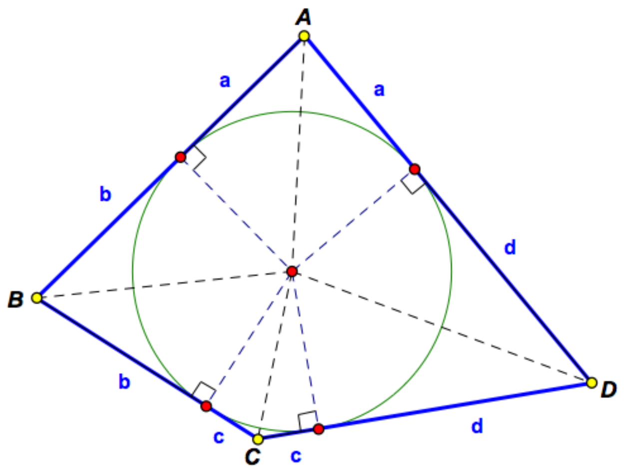

FIGURE 3: Tangential quadrilateral

Pitot's Theorem[8]:

A quadrilateral $ABCD$ is tangential if and only if $AB + CD = BC + DA$

Proof:

The proof of the forward implication is quite straightforward. Assume $ABCD$ is tangential as illustrated in Figure 3. Then labeling equal tangents from the vertices to the circle as indicated it follows that:

$$AB + CD = a + b + c + d = BC + DA$$

A ready-made geometry sketch for learners to dynamically explore and gain confidence in the validity of the converse before dealing with the proof is available at:

http://dynamicmathematicslearning.com/tangent-quad-converse.html

The proof of the converse, like its cyclic counterpart, is probably most easily done via proof by contradiction as follows. Consider Figure 4 where it is given that $AB + CD = BC + DA$. Construct the angle bisectors of angles A and B, and from their point of intersection I drop perpendiculars to sides AB, BC and AD. From I as centre, construct a circle through the feet of these perpendiculars so that sides AB, BC and AD are tangents to it. Assuming CD is not a tangent to this circle, construct a tangent CD' as shown with D' on AD (or its extension).

[8] This theorem is named after a French engineer Henri Pitot (1695-1771) who proved the forward implication in 1725. The converse was proved by the Swiss mathematician Jakob Steiner (1796-1863) in 1846.

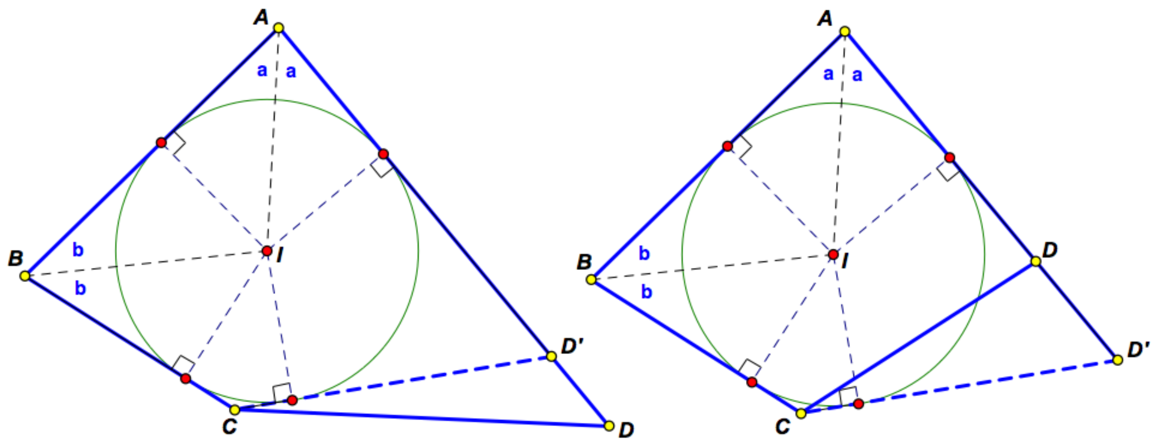

FIGURE 4: Proof of converse of Pitot's theorem

We now have $AB + CD' = BC + AD'$ since $ABCD'$ is tangential. But we are given that $AB + CD = BC + DA$ and therefore $CD' - CD = AD' - AD$ or $CD = CD' - AD' + AD$. However, in the first diagram in Figure 4, $AD = AD' + DD'$ and therefore $CD = CD' - AD' + (AD' + DD') = CD' + DD'$. This is impossible from the triangle inequality unless $DD' = 0$ and D' coincides with D, which contradicts our initial assumption that CD is not a tangent to the circle. It is left to the reader to check that the same contradiction applies to the 2nd case shown in Figure 4. QED.

It should be noted that a tangential quadrilateral could also be concave as shown in Figure 5 (in which case the extension of two of the sides are tangents to the incircle). A dynamic geometry sketch of a tangential quadrilateral, which can be dragged to become concave, is available to explore at:

http://dynamicmathematicslearning.com/tangential-quad.html

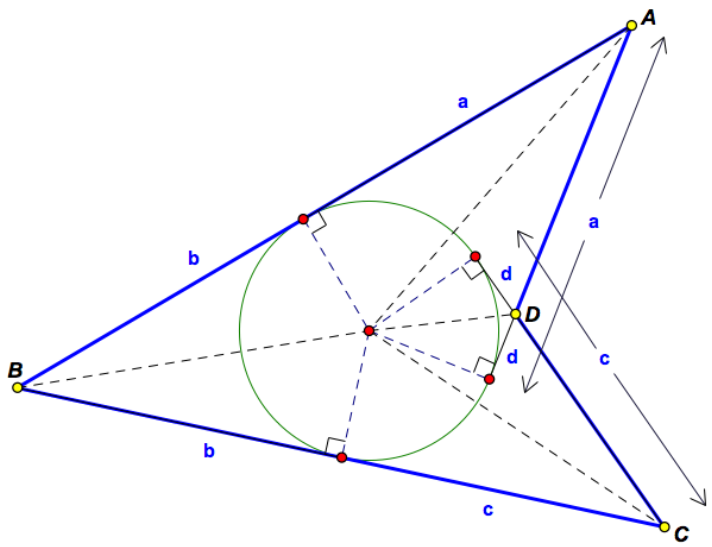

FIGURE 5: Concave tangential quadrilateral

The proof of the concave case shown in Figure 5 is also quite straight forward, since in this case we have $AB + CD = a + b + c - d = BC + DA$. The converse of the concave case can also be proved using proof by contradiction, and is left to the reader.

INCIRCLES

The following result is quite remarkable as it appears to have been discovered only quite recently. A dynamic sketch for learners to first investigate and conjecture the result is available at:

http://dynamicmathematicslearning.com/tangent-incircles-investigate.html

Theorem of Gusić & Mladinić

A quadrilateral is tangential if and only if the incircles of the two triangles formed by a diagonal are tangential to each other[9].

Proof:

If the incircles of triangles ABD and BCD of a quadrilateral $ABCD$ are tangential to one another as shown in Figure 6, then it follows as before that $AB + CD = a + b + c + d = BC + DA$. Therefore, according to Pitot's theorem, $ABCD$ is a tangential quadrilateral. The same argument applies if incircles drawn in the two triangles formed by the other diagonal are tangential to one another.

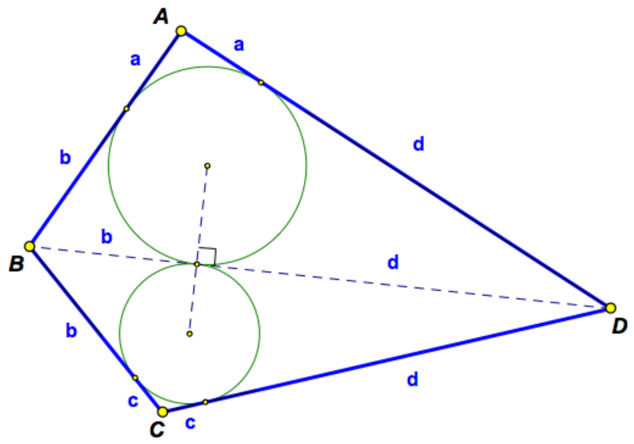

FIGURE 6: Tangential incircles of triangles ABD and BCD

To prove the converse, we shall first prove the following useful general theorem for any convex or concave quadrilateral (and which is also dynamically illustrated in the previous link).

Theorem 3: If a quadrilateral $ABCD$ is divided by a diagonal and the incircles of the two formed triangles are constructed, then the distance k between the two tangent points of the incircles to the diagonal is equal to $|(AB + CD) - (BC + DA)|/2$.

Proof:

Consider Figure 7 which shows a general (convex) quadrilateral with the incircles of triangles ABD and BCD constructed. Labeling the equal tangents to the circles as before, we have that:

$$\frac{|(AB + CD) - (BC + DA)|}{2} = \frac{|(a + b + c + d) - (b + k + c + d + k + a)|}{2} = \frac{2k}{2} = k$$

The concave case is left to the reader.

[9] This theorem is named after two Croatian colleagues, Jelena Gusić and Petar Mladinić (2001), who as far as I've been able to ascertain, have priority in first publishing the result in 2001 in a journal *Poučak*. Later publications by Worrall (2004) and Josefsson (2011) also mention and prove the theorem.

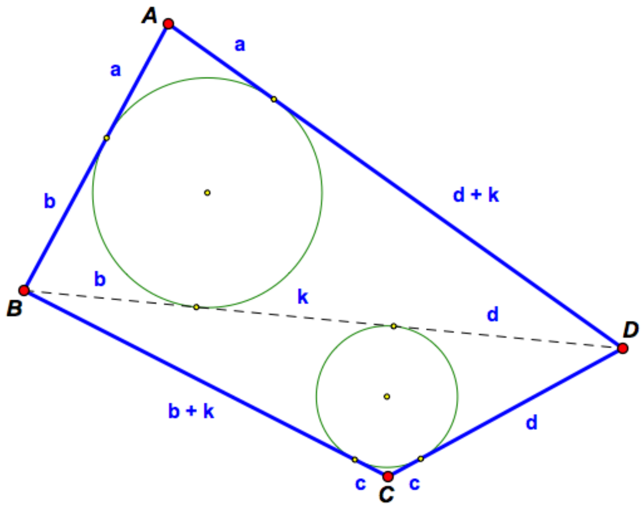

FIGURE 7: Incircles of triangles ABD and BCD

An obvious corollary to Theorem 3 is that the distance between the two tangent points of the incircles of triangles ABC and ACD to the diagonal AC is also equal to k. In addition, from Theorem 3, it now immediately follows that if $ABCD$ is a tangential quadrilateral, then we know from Pitot's Theorem that $AB + CD = BC + DA$, in which case the distance between the two incircles becomes zero, i.e. $k = 0$. In other words, the two incircles are tangential to one another. This then completes the proof of the converse of the Theorem of Gusić & Mladinić.

FURTHER APPLICATION

A neat little application of Theorem 3 is the following result, which can be dynamically explored at: http://dynamicmathematicslearning.com/tangent-hex-apply.html

Theorem 4: If a tangential hexagon $ABCDEF$ is triangulated by drawing three diagonals from any of its vertices, and the incircles of the four formed triangles are constructed, then the distance between the two tangent points of the incircles to the first diagonal is equal to the distance between the two tangent points of the incircles to the third diagonal.

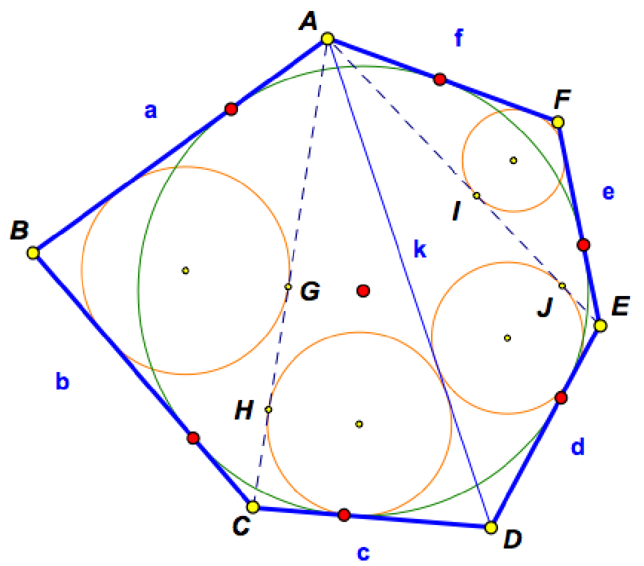

FIGURE 8: A triangulated tangential hexagon with incircles

Proof:

Consider Figure 8 which shows a tangential hexagon $ABCDEF$ with three diagonals drawn from vertex A to triangulate the hexagon. Label the sides of the hexagon consecutively by a, b, c, etc. and the main diagonal AD as k. We shall now use the following generalization of Pitot's theorem proved in De Villiers (1993, 2009), and which follows easily from the equal tangents to a circle theorem: "If a hexagon is tangential, then the two sums of its *alternate* sides are equal."[10]

From the aforementioned theorem, we have $a + c + e = b + d + f$. Next we apply Theorem 3 to $ABCD$ and $DEFA$, and use the alternate side sum result to obtain:

$$2GH = |(a + c) - (b + k)| = |(-e + d + f) - k| = |(d + f) - (e + k)| = 2IJ$$

$$\Rightarrow \quad GH = IJ$$

Concluding remarks

This paper has hopefully given the reader some taste of the mathematical possibilities of exploring tangential quadrilaterals. It provides a rich context for revising and applying geometric ideas and theorems such as equidistance, angle bisectors, incircle, incentre, and the equal tangents theorem, as well as the important proof technique of proof by contradiction. Many more beautiful results can be explored and proved regarding tangential quadrilaterals, some of which are given in the references.

Using dynamic geometry software, learners can first experimentally explore several of these results, formulating, checking and disproving conjectures, before engaging in the process of proof. However, some of these results, like the forward implication of Pitot's theorem, could also be meaningfully used to illustrate to learners the *discovery* function of proof, without any prior experimental investigation, by directly applying the equal tangents theorem to a tangential quadrilateral (compare De Villiers, 2003, pp. 68-69).

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[10] Note that this generalization is true for any tangential $2n$-gon, and what is normally labeled as 'opposite' sides of a tangential quadrilateral can also be regarded as its 'alternate' sides.

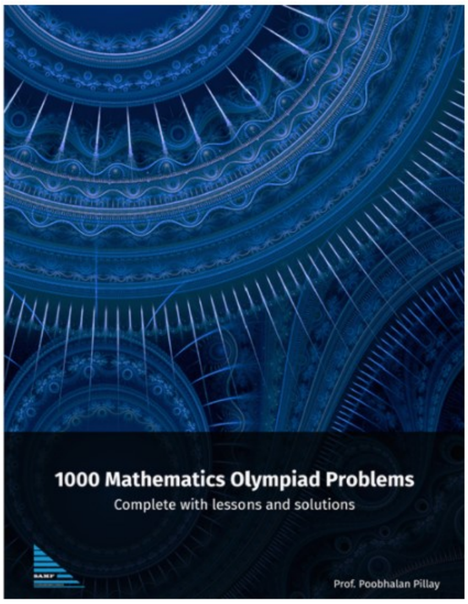
Book Review

1000 Mathematics Olympiad Problems
Poobhalan Pillay
South African Mathematics Foundation (SAMF), 2020
ISBN: 978-1-920080-50-1 (eBook, pp. 637)

Reviewed by Michael de Villiers[11]

> *"... a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to hidden truths, and ultimately a reminder of our pleasure in the successful solution."* - David Hilbert (1900)

This enticing set of Olympiad problems provides a glimpse for talented mathematics learners into the exciting world of research mathematics that is seldom accomplished in the usual classroom situation. A good Olympiad problem captures in miniature the exhilarating process of creating mathematics. All the aspects of creativity are there: the period of immersion in the problem, the examination of possible approaches, and the pursuit of various paths to solution. There is the fruitless dead end, as well as the path that ends abruptly but offers new perspectives, leading eventually to the discovery of a better route. Most obviously, grappling with a tough problem provides useful practice in dealing with the frustration of working at material that refuses to yield. Finally, there is hopefully the flash of insight that finally heralds the start of a successful approach.



The book *1000 Mathematics Olympiad Problems* is, however, much more than just another collection of interesting, challenging problems, but is instead organised specifically for individual learning as well as for teaching problem solving. The problems are conveniently categorized into different topics and graded into different cognitive levels of difficulty so that learners and teachers will have some idea beforehand of the 'accessibility' of individual problems. While the vast majority of the problems in this book have their origins in the annual South African Mathematics Olympiad (SAMO), the SAMF's Team Competitions, and SAMF's Talent Searches, other resources on problem solving have also been used. Learners are encouraged to first attempt to solve the problems, and if successful to try to solve them in more than one way. Fully worked solutions to all the problems are provided at the back of the book.

[11] Michael de Villiers is the coordinator of the Senior Round 1&2 Committee of the South African Mathematics Olympiad (SAMO).

While some problems in this book were new to me, others I haven't seen for years, and it was wonderful to be re-acquainted with some of those. In more than one instance, a particularly attractive problem caught my eye and had me reaching for pen and paper, so irresistible was the urge to wrestle with and conquer the offered challenge. The number, variety and range of problems is staggering, and it was clearly a Herculean effort to select, compile and present these together. Prof. Pillay has done the South African mathematics community a huge service with this massive labour of love, which is likely to have a hugely positive impact on the development of young mathematical talent in our country for many years to come.

In addition to the tantalizing 1000 problems, the book includes eleven lessons and seven appendices. To give readers a flavour of the scope and breadth of these, here are a few examples of the wide range of topics covered:

- Numbers: All numbers, from natural numbers to complex numbers are discussed in detail.
- Sequences & series: Arithmetic and geometric sequences and series, triangular numbers, Fibonacci sequences, Farey sequences, sums of the first n squares and cubes, proof by mathematical induction.
- Rational and irrational numbers: Irrationality proofs, recurring decimals, Farey sequences.
- Factorisation: Factorisation of sums and difference of integral powers.
- Geometry: general polygons, concurrency theorems, Euler's formula and Platonic solids.
- Areas: Heron's formula, areas of regular polygons, Pick's Theorem.
- Trigonometry: Trigonometric ratios of multiples of 18°, sum of sines.
- Counting and Probability: Binomial coefficients, Binomial Theorem, counting with repetitions.

Topics in elementary number theory include congruence arithmetic, the Euclidean algorithm, the Chinese Remainder Theorem, solutions of Diophantine Equations, Fermat's Little theorem, perfect numbers, and Mersenne and Fermat primes. Advanced topics in geometry include the Euler line, the nine-point circle, the power of a point, Ceva's Theorem, Ptolemy's Theorem, and a detailed treatment of the regular pentagon. The Cauchy-Schwartz and rearrangement inequalities, as well as the inter-relationships amongst the difference means are discussed. The lesson on polynomials provides an in-depth treatment of quadratic polynomials, while moving onto a general discussion of roots of arbitrary polynomials over complex numbers, and consequences of the Fundamental Theorem of Algebra.

The book often weaves together related problems, so that insights gradually become techniques, tricks slowly become methods, and methods eventually evolve into mastery. While each individual problem may be a delectable appetiser on its own, the table is set in this book for a more satisfying fare, which will take the reader deeper into mathematics than might any single problem or contest.

This book is written by an experienced and well-known mathematical problem solver, who served on the SAMO committee for several years, and is currently the academic coordinator of the Siyanqoba Problem Solving Training Programme. The book is aimed at talented high school learners and their teachers, as well as beginning university students and instructors. It can be used as a text for advanced problem-solving courses, for self-study, or as a resource for teachers and learners training for mathematical competitions and for teacher professional development, seminars, and workshops.

1000 Mathematics Olympiad Problems is strongly recommended for anyone interested in creative problem solving in mathematics, and is available for R520 (VAT inclusive) at https://www.samf.ac.za/en/ as a downloadable PDF. It has already taken up a prized position in my personal library, and is bound to continue to provide me with many hours of intellectual pleasure.

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Suggestions to writers

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