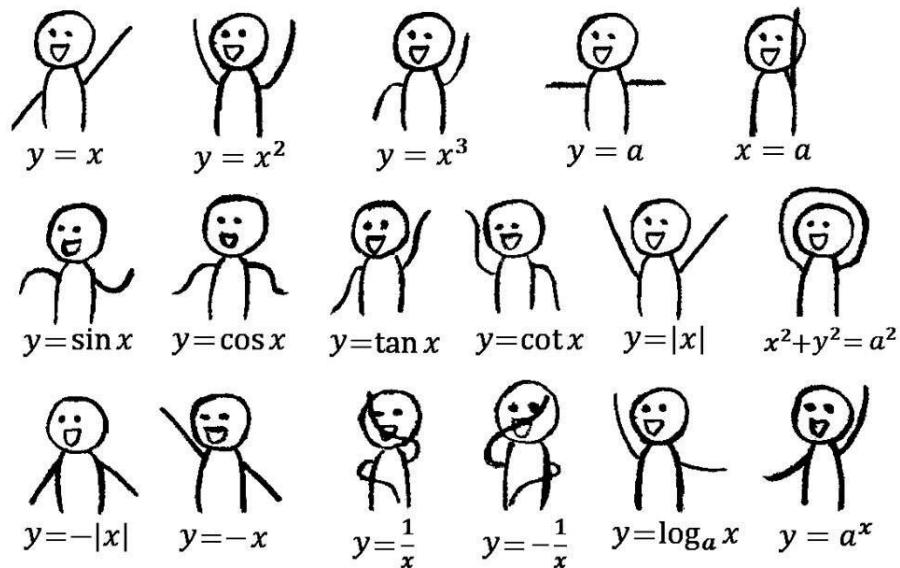


# Education, Research & Development

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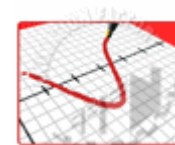
University of Zagreb  
Faculty of Geodesy



## The Investigation of the concept of function according to van Hiele's levels

Nikol Radović, MSc

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- In the seventh grade of elementary school (according to the Croatian curriculum), students learn the concept of function for the first time, and throughout school they deepen and practice it by solving various problems tasks.
- However, the question is how our students recognize the concepts (functions) they have learned in their environment, in some classical tasks, used to measure van Hiele's levels of acquisition.
- It is known from the literature (van Hiele, 1986.; Teppo, 1991.; Mason, 2009.; Jurgenson et al, 2020.), that van Hiele's levels are used to assess geometry, but it is also possible to apply them to other areas of mathematics.
- The same is true for functions (Ishodi, 1987) and trigonometry (Walshu, 2015).



- The aim of the investigation conducted within the framework of the project „*Van Hiele Level of Mathematical Achievement of Students in the Republic of Croatia*” approved by the Ministry of Science and Education of the Republic of Croatia and carried out in cooperation with the University of Zadar, is to verify the adoption of the concept.
- Level **I** – everyday language,
- Level **II** – arithmetic,
- Level **III** – algebra and geometry,
- Level **IV** – accounting,
- Level **V** – mathematical analysis.

- The investigation was conducted over two years on elementary and high school students.
- The first test was conducted in May 2019 in 10 high schools and i 5 elementary schools.
- 1181 high school students and 155 elementary school students participated in the tests (Osijek, Vukovar, Vinkovci, Zagreb, Metković, Split, Zadar, Rijeka and Pula).
- The second testing phase lasted from October 2020 to January 2021 during the pandemic.
- A total of 1439 students were tested, of which 148 students were in elementary school.
- In the second testing year, the same students were tested to check their progress.

- For testing purposes, a group of teachers has developed tests for linear, quadratic, exponential, and logarithmic functions.
- These functions (with trigonometric functions) are the basis for the application of mathematics in the science (physics, biology and chemistry).



HUNI - Croatian Association of Teacher Researchers

Udruga i mreža nastavnika i istraživača iz područja nastave



## LINEARNA FUNKCIJA

## LINEAR FUNCTION

(škola)

(SCHOOL)

Sponzori projekta



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## KVADRATNA FUNKCIJA

(škola)

Sponzori projekta



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## EKSPONENCIJALNA FUNKCIJA

(škola)

Sponzori projekta



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- Each test has 9 questions – 3 questions for each level
- It should be emphasized that for each functional group the level was tested from 1 to 3 according to van Hiele .

Level I – everyday language,

Level II – arithmetic,

Level III – algebra and geometry,

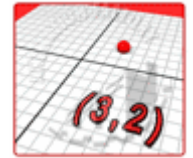
- Example\_*Test Linear Function*



# Linear function

## Linearna funkcija

Match the expression from the left column with corresponding description from the right column



1. Povežite izraz iz lijevog stupca s odgovarajućim opisom iz desnog .

A given number is increased by three

određeni broj uvećan je za tri

$$3x+2$$

A given number is increased three times

određeni broj uvećan je tri puta

$$x+3$$

Double the value of given number is increased by three

dvostruka vrijednost određenog broja uvećana je za tri

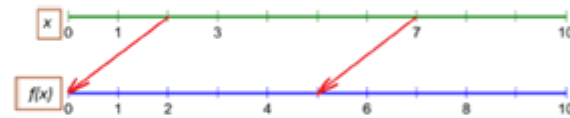
$$2x+3$$

$$2x+2$$

$$3x$$

2. Proučite pridruživanje na brojevnim pravcima. Od zadanog broja oduzmite 2, popunite tablicu i prikažite pridruživanje na brojevnim pravcima.

$x$	$f(x)$
2	0
4	
	5
	7



3. Dana se mreža može koristiti za šifriranje pa je tako uređenom paru (1, 5) pridruženo slovo E, a broj 7 pridružen je uređenom paru (3, 0).

5	F	E	D	C	B	A
4	G	H	I	J	K	L
3	R	Q	P	O	N	M
2	S	T	U	V	W	X
1	3	2	1	0	Z	Y
0	4	5	6	7	8	9
	0	1	2	3	4	5

a) Odredite šifru za riječ SUNCE.

b) Dešifrirajte izjavu

(3, 2) (3, 3) (5, 4) (2, 4) (5, 3) (2, 3) (5, 4) (1, 5) (0, 2).

## Linearna funkcija

1. Povežite izraz iz lijevog stupca s odgovarajućim opisom iz desn

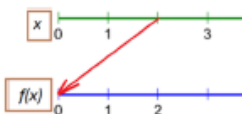
određeni broj uvećan je za tri

određeni broj uvećan je tri puta

dvostruka vrijednost određenog broja uvećana je za tri

2. Proučite pridruživanje na brojevnim pravcima. Od zadanog broj pridruživanje na brojevnim pravcima.

$x$	$f(x)$
2	0
4	
	5
	7



3. Dana se mreža može koristiti za šifriranje pa je tako uređenom pridružen je uređenom paru (3, 0).

5	F	E	D	C	B	A
4	G	H	I	J	K	L
3	R	Q	P	O	N	M
2	S	T	U	V	W	X
1	3	2	1	0	Z	Y
0	4	5	6	7	8	9
	0	1	2	3	4	5

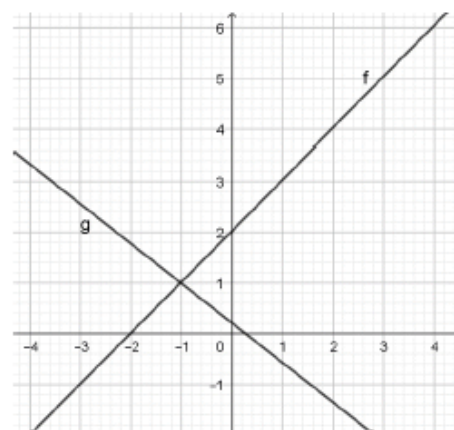
a) Odredite šifru za riječ :

b) Dešifrirajte izjavu (3, 2) (3, 3) (5, 4) (2, 4) (5

4. a) Prikažite grafički funkciju  $f(x) = -\frac{1}{3}x + 2$ .  
b) Odredite grafički i računski nultočku funkcije  $f$ .

5. Na slici su prikazani grafovi linearnih funkcija  $f$  i  $g$ .

a) Poredajte vrijednosti  $f(2)$ ,  $g(0)$  i  $g(25)$  od najmanje do najveće



b) Postoji li  $x$  za koje je  $f(x) = g(x)$ ? Ako postoji napišite koji je  $t$

6. Ana je odlučila za praznike jedan tjedan svakodnevno mjeriti koliko naraste trava. Dobiveni su rezultati prikazani u tablici:

Broj dana	0	1	2	3	4	5	6	7
Visina trave (mm)	10	12	14	16	18	20	22	24

a) Nacrtajte koordinatni sustav tako da na osi  $x$  označite dane, a na osi  $y$  visinu trave (u mm) te ucrtajte točke iz gornje tablice vrijednosti. Spojite ih. Koju funkciju predstavlja dobiveni grafički prikaz?



b) Visina trave mijenja se po danima i to se može opisati funkcijom  $f$ . Odredite pravilo pridruživanja  $f(x)$ .

c) Koje su koordinate točke presjeka grafa funkcije  $f$  i  $y$ -osi? Koje značenje ima ta točka?

d) Odredite visinu trave nakon 14 dana ako se nije kosila dotad.

e) Je li funkcija  $f$  rastuća ili padajuća?

7. Na slici su prikazane cijene pića. Cijena pića je linearna funkcija



a) Kolika je cijena

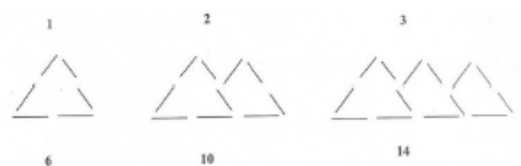
b) Odredite funkciju ovisno o broju mililitara

9. Teta Marina ima problem sa sudoperom. Naime, "zaštopan" je već tjedan dana pa se ona raspitala o cijenama rada dvojice najboljih vodoinstalatera u gradu. Saznala je tako da majstor Marko naplaćuje svoj dolazak 50 kn i 36 kn po satu rada, dok majstor Stipe uzima 35 kn za dolazak u kuću i 39 kn po satu rada. Teta Marina je neodlučna, pomozite joj.

a) Odredite algebarske prikaze funkcija koji prikazuju ovisnost cijene o trajanju popravka (u satima) za oba majstora.

b) Ako je mali kvar koji se može popraviti za tri sata, isplativije je zvati majstora \_\_\_\_\_, ali ako je potrebno šest sati rada više se isplati pozvati \_\_\_\_\_. Za \_\_\_\_\_ sati rada obojica će majstora naplatiti jednaki iznos od \_\_\_\_\_ kn.

8. Promotrite sljedeći niz oblika napravljenih od šibica.



a) Nacrtajte sljedeći, četvrti, oblik u nizu.

b) Odredite broj šibica u petom obliku u nizu.

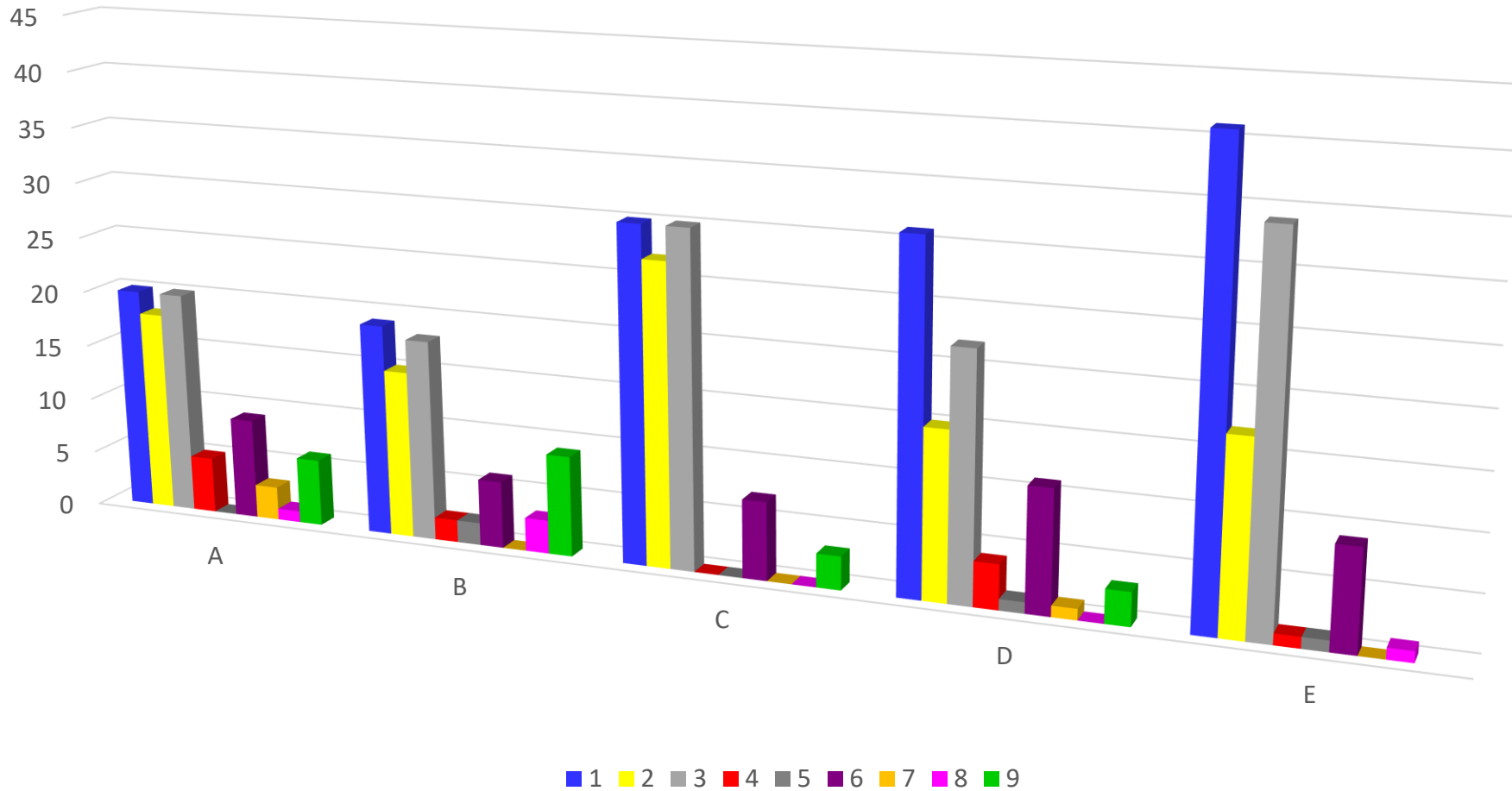
c) Ivan ima 300 šibica. On njima želi prikazati niz oblika kako j zadnjem obliku koji će Ivan moći prikazati? Koliko će mu šibic

- After the first test, the tests for each functional group were corrected, refined, and supplemented according to the experiences and comments of the teachers who wanted to test their students.
- The results were unexpected (even worse).



# Results:

## Linear function\_elementary school

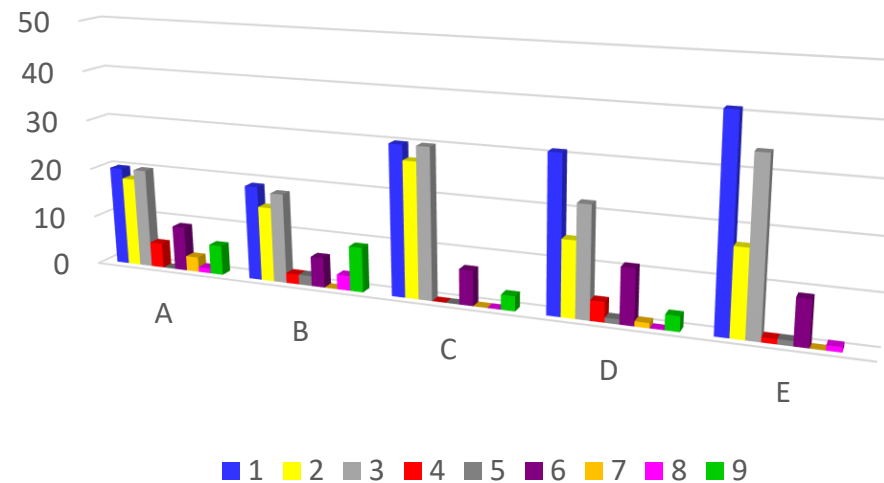


6. Ana je odlučila za praznike jedan tjedan svakodnevno mjeriti koliko naraste trava. Dobiveni su rezultati prikazani u tablici:

Broj dana	0	1	2	3	4	5	6	7
Visina trave (mm)	10	12	14	16	18	20	22	24

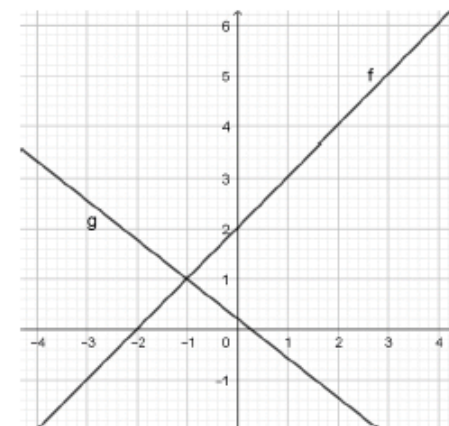
a) Nacrtajte koordinatni sustav tako da na osi  $x$  označite dane, a na osi  $y$  visinu trave (u mm) te ucrtajte točke iz gornje tablice vrijednosti. Spojite ih. Koju funkciju predstavlja dobiveni grafički prikaz?

Wrong conclusion - the function is increasing, not linear



5. Na slici su prikazani grafovi linearnih funkcija  $f$  i  $g$ .

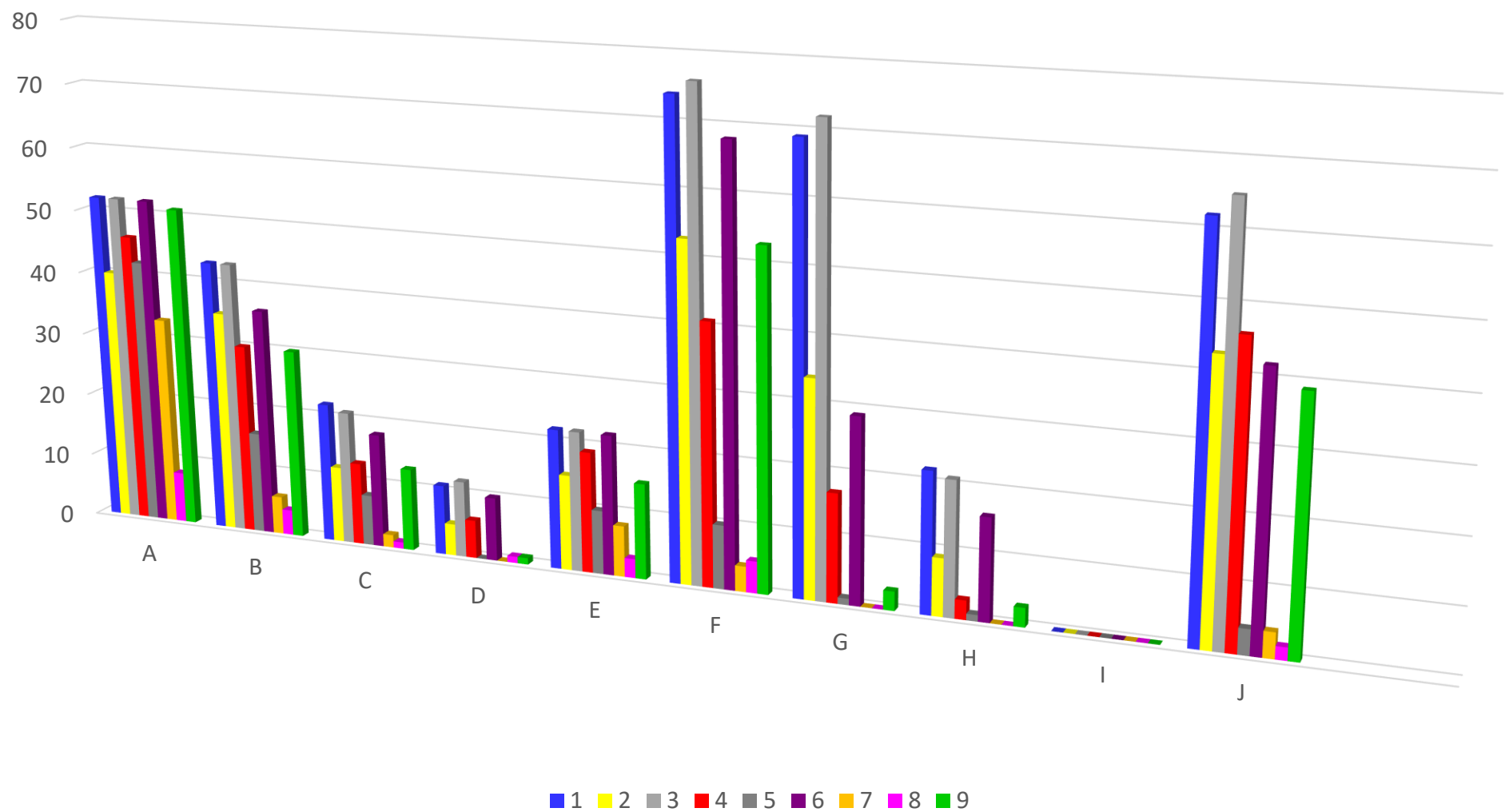
a) Poredajte vrijednosti  $f(2)$ ,  $g(0)$  i  $g(25)$  od najmanje do najveće.



b) Postoji li  $x$  za koje je  $f(x) = g(x)$ ? Ako postoji napišite koji je to  $x$  i koliki je njemu pridruženi  $f(x)$ ?

they do not understand the meaning of  $f(x) = g(x)$ ,  $x = ?$ ,  
neither  $g(25) < g(0) < f(2)$

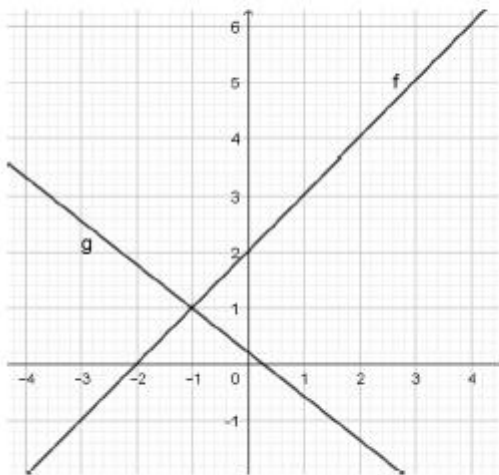
# Linear function\_high school



# Linear function\_high school

5. Na slici su prikazani grafovi linearnih funkcija  $f$  i  $g$ .

a) Poredajte vrijednosti  $f(2)$ ,  $g(0)$  i  $g(25)$  od najmanje do najveće.



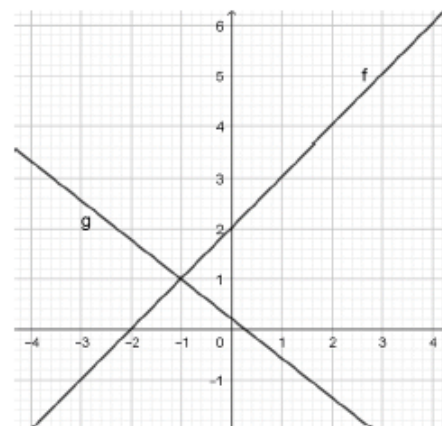
b) Postoji li  $x$  za koje je  $f(x) = g(x)$ ? Ako postoji napišite koji je to  $x$  i koliki je njemu pridruženi  $f(x)$ ?

they don't know  $f(x) = g(x)$  means

they don't know/ understand how to  
determine  $x$  graphically from  $f(x) = g(x)$

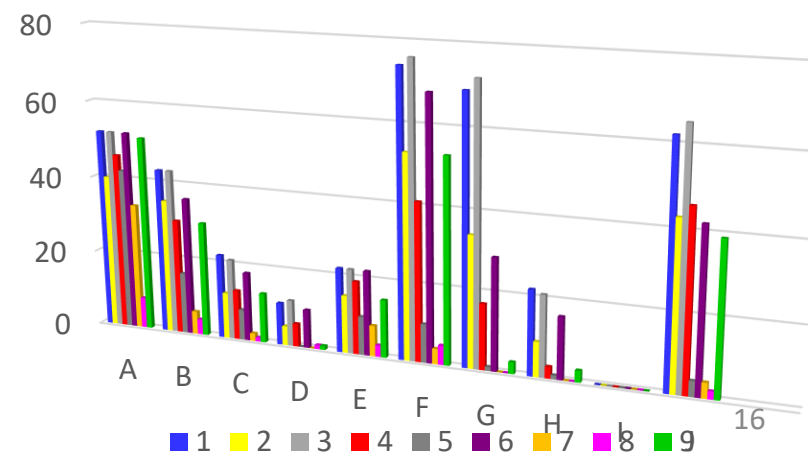
5. Na slici su prikazani grafovi linearnih funkcija  $f$  i  $g$ .

a) Poredajte vrijednosti  $f(2)$ ,  $g(0)$  i  $g(25)$  od najmanje do najveće.



b) Postoji li  $x$  za koje je  $f(x) = g(x)$ ? Ako postoji napišite koji je to  $x$  i koliki je njemu pridruženi  $f(x)$ ?

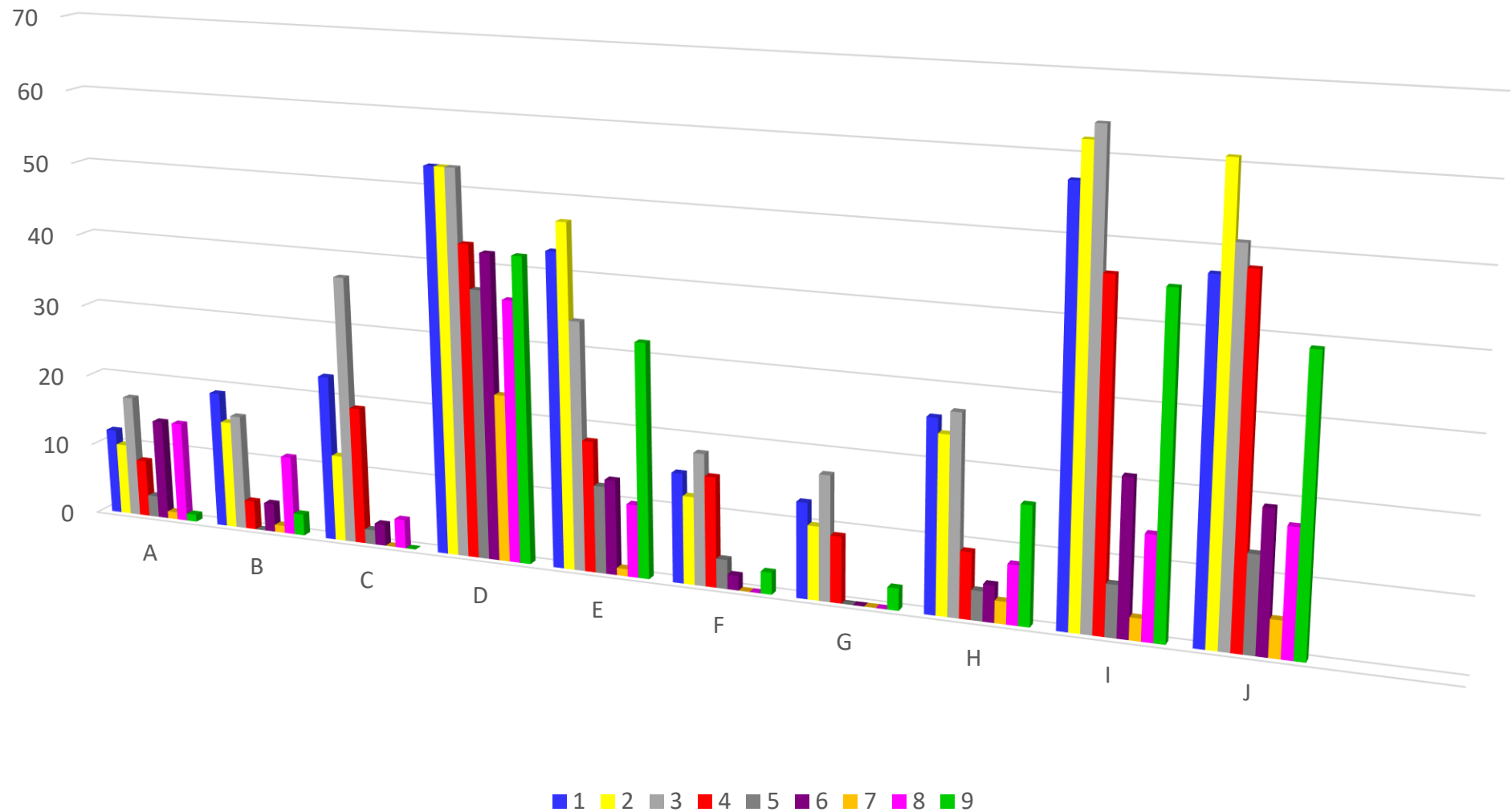
exchange the first and second coordinates.  
instead of  $(0, 2)$  they write  $(2, 0)$



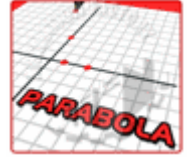
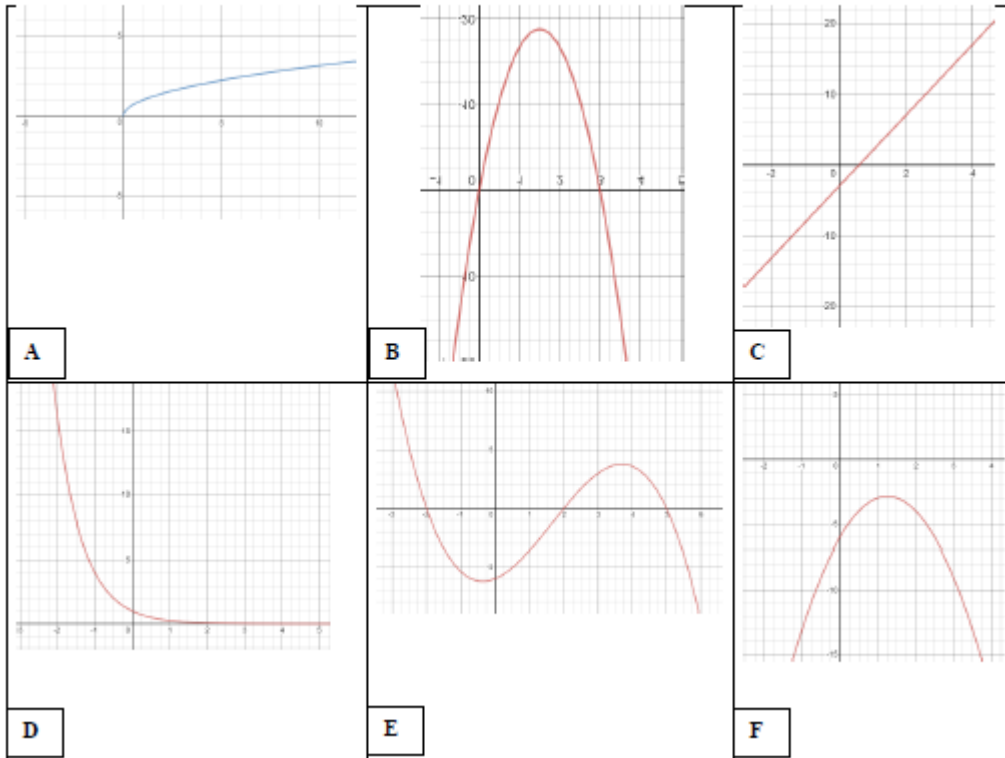


# Quadratic function\_high school

-behind letters A, B, . . . – are not the same school as at linear function



3. Koje od ponuđenih slika prikazuju graf kvadratne funkcije. Obrazložite svoj odabir.



Graph B is drawn incorrectly, so it is not a parabola.

Graph E is a parabola?!

Graph B is parabola, but coordinate system is wrong.

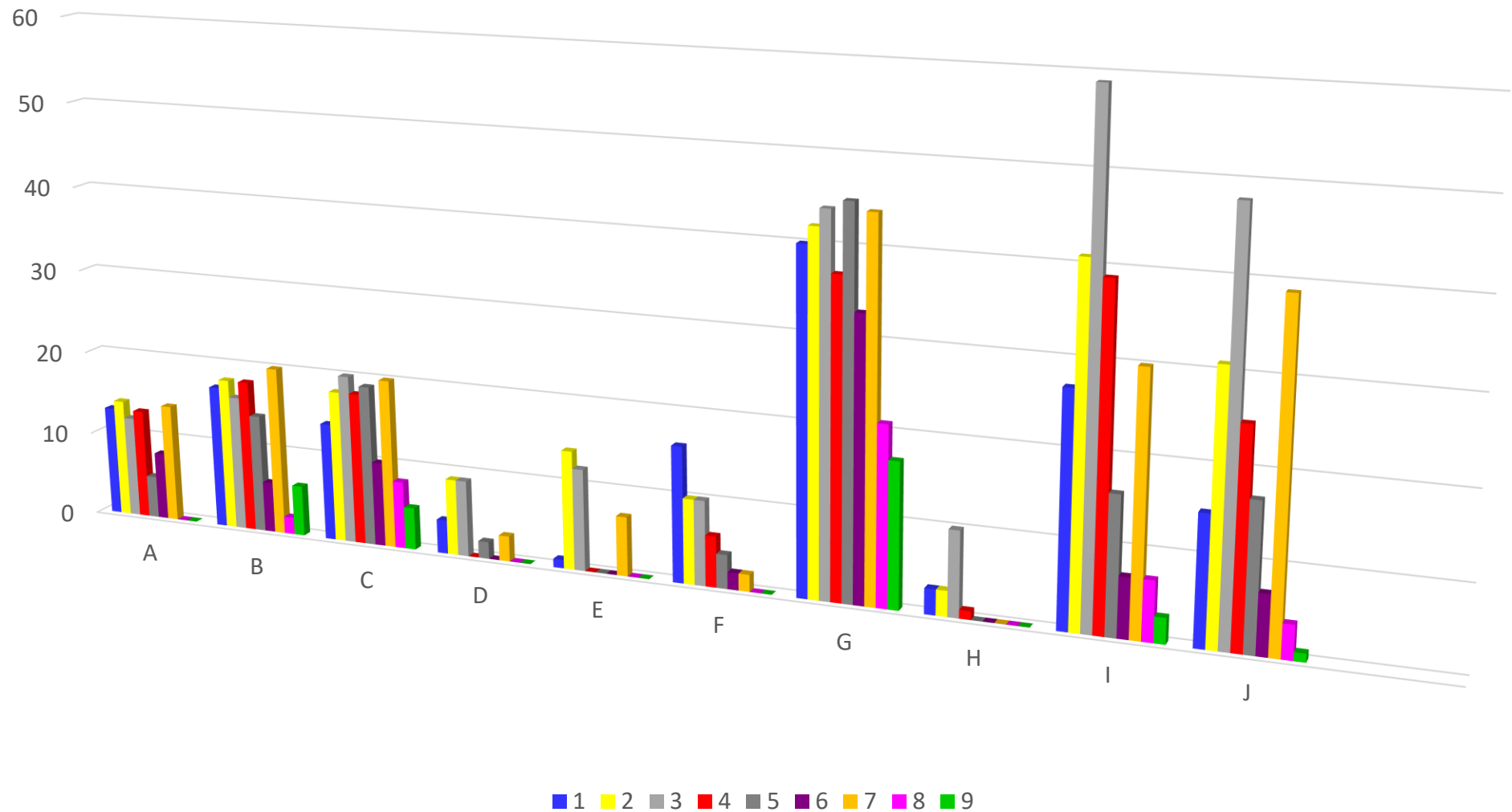
4. Dana je funkcija  $f(x) = 4x^2 + 12x + 14$ . Postupkom svođenja na potpuni kvadrat funkcija se može zapisati i u obliku:  $f(x) = 4\left(x + \frac{3}{2}\right)^2 + 5$ .

Kojim od navedenih oblika ćemo jednostavnije odrediti koordinate tjemena? Obrazložite odgovor.

They do not know the vertex shape of function and they do not know how to find the coordinates

# Exponetial function\_high school

behind letters A, B, . . . – are not the same school as at linear/quadratic function



- In working through these tests, it was found that mathematics classes or mathematics high school have a more durable knowledge and much greater number of correctly solved problems than classes in other high schools (general education, language, ...)

- For example, two mathematics classes in the same high school in terms of the overall result in that high school.

*Quadratic function - 19 students in class*

LEVEL	I.	II.	III.
NUMBER OF TASKS	57	57	57
NUMBER OF SOLVED TASKS	$16+18+11=45$	$14+7+12=33$	$3+10+14=27$
PERCENTAGE OF SOLVED TASKS IN THE CLASS	$\frac{45}{57} = 78,9\%$	$\frac{33}{57} = 57,9\%$	$\frac{27}{57} = 47,4\%$
PERCENTAGE OF SOLVED TASKS IN THE SCHOOL	70,7%	35,6%	23,6%

*Exponential function - 15 students in class*

LEVEL	I.	II.	III.
NUMBER OF TASKS	45	45	45
NUMBER OF SOLVED TASKS	$6+11+13=30$	$13+11+5=29$	$15+4+1=20$
PERCENTAGE OF SOLVED TASKS IN THE CLASS	$\frac{30}{45} = 66,7\%$	$\frac{29}{45} = 64,4\%$	$\frac{20}{45} = 44,4\%$
PERCENTAGE OF SOLVED TASKS IN THE SCHOOL	42,9%	22,4%	20,1%

- Taking at least the correct solution of **two** of the three tasks as a criterion for success.
- It can be seen that the students in th mathematical class are at Level I and **close** to Level II in their knowledge of the **quadratic function**. The overall score of the students in this school is at Level I, but far from the Levels II and III.
- The students in this mathematical class are at Level I and **near** Level II in their knowledge of the **exponential function**. The overall result of the students of this school is **far** from Levels I, II and III.

- Second test – postponed because the pandemic
- The results are devastating,

Linear function - elementary school

145 students

LEVEL I TASKS: 1. - 3.	0:	13 students hasn't solved any of the first three tasks
	1:	33 students solved <b>only one</b> of the first three tasks
	2:	33 students solved two of the first three tasks
	3:	66 students have solved <b>all three tasks</b> of the first three tasks
LEVEL II TASKS: 4. - 6.	0:	102 students hasn't solved any of the 4th to 6th tasks
	1:	22 students solved <b>only one</b> of the 4th to 6th tasks
	2:	17 students solved two of the 4th to 6th tasks
	3:	4 students have solved all three tasks of the 4th to 6th tasks
LEVEL III TASKS: 7. - 9.	0:	132 students hasn't solved any of the 7th to 9th tasks
	1:	10 students solved only one of the 7th to 9th tasks
	2:	2 students solved two of the 7th to 9th tasks
	3:	1 student solved all three tasks of the 7th to 9th tasks

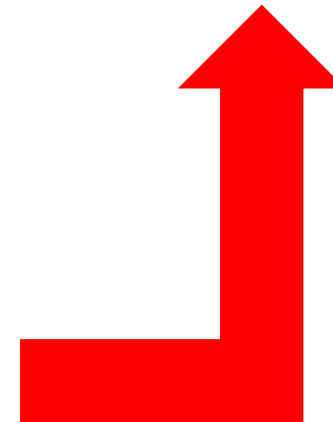
Most students are **only at the Level I**, far from the theoretically predicted Level III



Linear function - high school 497 students

LEVEL I TASKS: 1. - 3.	0: 1: 2: 3:	21 students hasn't solved any of the first three tasks 99 students solved only one of the first three tasks 106 students solved two of the first three tasks 171 students have solved all three tasks the first three tasks
LEVEL II TASKS: 4. - 6.	0: 1: 2: 3:	281 students hasn't solved any of the 4th to 6th tasks 114 students solved only one of the 4th to 6th tasks 67 students solved two of the 4th to 6th tasks 35 students have solved all three tasks of the 4th to 6th tasks
LEVEL III TASKS: 7. - 9.	0: 1: 2: 3:	401 students hasn't solved any of the 7th to 9th tasks 60 students solved only one of the 7th to 9th tasks 28 students solved two of the 7th to 9th tasks 2 students have solved all three tasks of the 7th to 9th tasks

The majority of students are at Level I, while 20 % are at the Level II and theoretically predict Level III is too far



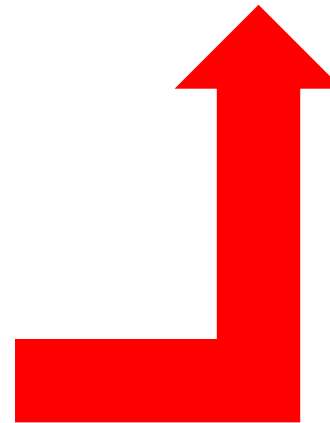


Quadratic function - high school

142 students

LEVEL I TASKS: 1. - 3.	0:	1 student haven't solved any of the first three tasks
	1:	17 students solved only one of the first three tasks
	2:	57 students solved two of the first three tasks
	3:	67 students have solved all three tasks the first three tasks
LEVEL II TASKS: 4. - 6.	0:	52 students hasn't solved any of the 4th to 6th tasks
	1:	32 students solved only one of the 4th to 6th tasks
	2:	36 students solved two of the 4th to 6th tasks
	3:	22 students have solved all three tasks of the 4th to 6th tasks
LEVEL III TASKS: 7. - 9.	0:	14 students hasn't solved any of the 7th to 9th tasks
	1:	54 students solved only one of the 7th to 9th tasks
	2:	50 students solved two of the 7th to 9th tasks
	3:	24 students have solved all three tasks of the 7th to 9th tasks

The majority of students are at Level I, while 41 % are at the Level II and 52 % are at the theoretically predict Level III

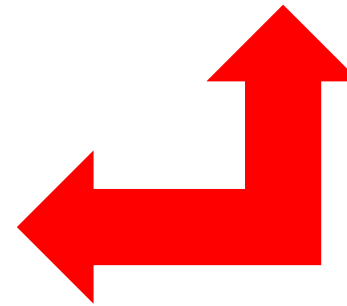


### Logarithmic function - high school... 214 students

LEVEL I TASKS: 1. - 3.	0: 1: 2: 3:	0 student haven't solved any of the first three tasks 5 students solved only one of the first three tasks 36 students solved two of the first three tasks 173 students have solved all three tasks the first three tasks
LEVEL II TASKS: 4. - 6.	0: 1: 2: 3:	99 students hasn't solved any of the 4th to 6th tasks 41 students solved <b>only one</b> of the 4th to 6th tasks 40 students solved two of the 4th to 6th tasks 34 students have solved all three tasks of the 4th to 6th tasks
LEVEL III TASKS: 7. - 9.	0: 1: 2: 3:	149 students hasn't solved any of the 7th to 9th tasks 42 students solved only one of the 7th to 9th tasks 13 students solved two of the 7th to 9th tasks 10 students have solved all three tasks of the 7th to 9th tasks

### Exponential function - high school... 155 students

LEVEL I TASKS: 1. - 3.	0: 1: 2: 3:	17 students hasn't solved any of the first three tasks 79 students solved only one of the first three tasks 52 students solved two of the first three tasks 7 students have solved all three tasks the first three tasks
LEVEL II TASKS: 4. - 6.	0: 1: 2: 3:	17 students hasn't solved any of the 4th to 6th tasks 17 students solved <b>only one</b> of the 4th to 6th tasks 66 students solved two of the 4th to 6th tasks 55 students have solved all three tasks of the 4th to 6th tasks
LEVEL III TASKS: 7. - 9.	0: 1: 2: 3:	35 students hasn't solved any of the 7th to 9th tasks 55 students solved only one of the 7th to 9th tasks 39 students solved two of the 7th to 9th tasks 28 students have solved all three tasks of the 7th to 9th tasks



## EXAMPLE.

The price of the drinks is 7 HRK (0.93 euros) for 100 ml, 12 HRK (1.6 euros) for 125 ml and 17 HRK (2.27 euros) for 150 ml. What is the price of a 200 ml drink?

- This is a linear function problem from the first test.
- It was solved by less than 3 % of the students in the 8th grade (elementary school), and in the 1st grade of high school (without mathematics subjects, which have a larger number of hours of mathematics) this problem was solves only by 6 % and in „mathematical” classes by 8 %.
- This task was intended (because of the possible application and use of the properties of the linear function) tor the Level III
- On closer inspection, it doesn't even require „spetial” calculations and mathematical knowledge and can be solved by rote.
- Thus, our students weren't prepared for „sophisticated” application of knowledge about the linear function, nor for possible „ingenuity” in solving task.



- The reasons for **poor** knowledge of functions are many, but it is obvious that teaching in the Republic of Croatia does not "produce" a good result.
- From these data it is evident that the content, tasks and teaching of mathematics in the Croatian school system should be modernized, as they do not produce good results.
- Continuation of this condition without the necessary measures will lead to similar or even worse results.
- This research on the knowledge of mathematical functions, as well as the results of the final exams and the international PISA and TIMS surveys among our students, clearly show that mathematics education needs to be redesigned, theoretically and professionally based on the scientific foundation and the experience of the best teachers and implemented consistently and dynamically.



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