

INVERSION AS A LANGUAGE FOR FLORAL VISUALIZATION

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Abstract: *Flowers captivate us with their colour and scent. Flowers and mathematics – can they be related? It is known that mathematics is the language of science. Does this include flora? In this paper, to answer these questions, we will show how mathematical and geometric transformations can be applied to the visualisation of flowers using a dynamic geometry programme as a powerful tool.*

Keywords: flower, inversion, petals, Sketchpad, visualisation.

1 INTRODUCTION

Plane transformations (translation, axial symmetry, rotation) map lines to lines and circles to circles. We are interested in studying mappings that map a set of lines and circles back to that set, where a line may be mapped either to a line or to a circle, known as inversion. Although this transformation is used to solve various geometric problems, particularly in the geometry of the circle, it will be a powerful tool for us, together with Sketchpad, for visualising various figures, especially flowers.

2 INVERSION: DEFINITION AND PROPERTIES

Inversion is a generalisation of the plane transformation known as axial symmetry or reflection.

2.1. Definition

A circle $k(O, r)$ is given in the plane. For every point T of the ray $\overrightarrow{OT}$ (except the centre O), the inversion with respect to the circle k is the point T' for which:

The points O , T , and T' are collinear,

The points T and T' are on the same side of the point O ,

$$|OT| \cdot |OT'| = r^2, T' \in \overrightarrow{OT}.$$

The point O is called the centre or inversion centre, and r^2 is the constant or inversion coefficient. The circle $k(O, r)$ is the inversion circle.

Every point on the inversion circle maps onto itself. Therefore, every point on the inversion circle is a fixed point, and these are the only fixed points. Specifically, if the point T is on the circle $k(O, r)$, then $|OT| = r$. From the property

$$|OT| \cdot |OT'| = r^2, T' \in \overrightarrow{OT},$$

Then it must also be true that $|OT'| = r$. Since the points T and T' are on the same side of point O , it follows that $T = T'$.

From this definition, it follows that points lying on a circle of radius r with centre at point O are mapped onto themselves. Points from the interior of the circle are mapped onto its exterior, and vice versa.

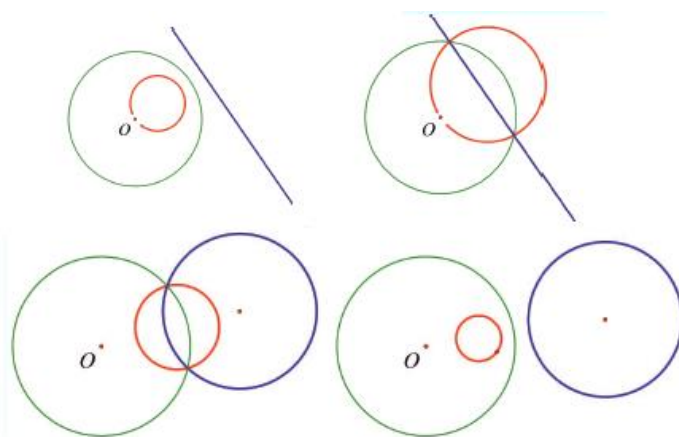


Figure 1: Basic properties of an Inversion with centre at point O .

Let us list the basic properties of an inversion with centre at point O . We distinguish between cases where a point is on a line or circle (incident) and where it is not (see Figure 1).

- A line containing point O , the centre of the inversion, is mapped to a line that also contains point O .
- A line that does not contain point O is mapped to a circle that contains point O .
- A circle that passes through the centre of the inversion is mapped to a line that does not pass through O .
- A circle that does not pass through point O is mapped to a circle that does not pass through point O .

2.2 Construction of the inverse image of a point

We distinguish two cases:

2.2.1. The point T is inside the circle $k(O, r)$.

Let us construct the inverse image T^* of the point T with respect to the circle of inversion $k(O, r)$.

Construction steps:

1. Draw the line x through points O and T .
2. Draw the perpendicular n through point T to the line x .
3. The line n intersects the circle k at points D and D_1 .
4. Construct the tangent t to the circle at point D .
5. The tangent t and the line x intersect at the required point T^* , as shown in Figure 2. The same result is obtained by constructing the tangent t' at point D_1 .

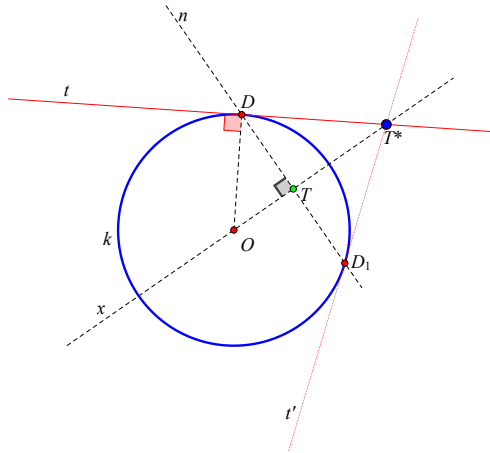


Figure 2: Construction of the inverse image of a point located inside the inversion circle.

Now we need to prove that the constructed point satisfies the required conditions. Since $|\angle OTD| = |\angle ODT^*| = 90^\circ$ and $|\angle TOD| = |\angle DOT^*| = 90^\circ$, it follows that the triangles $\triangle OTD$ and $\triangle ODT^*$ are congruent in all three angles, i.e. the triangles are similar. Therefore, we can calculate:

$$\frac{|OT^*|}{r} = \frac{r}{|OT|} \Rightarrow |OT| \cdot |OT^*| = r^2.$$

2.2.2. *The point T is outside the circle $k(O, r)$.*

We construct the inverse image T^* of point T with respect to the circle of inversion $k(O, r)$:

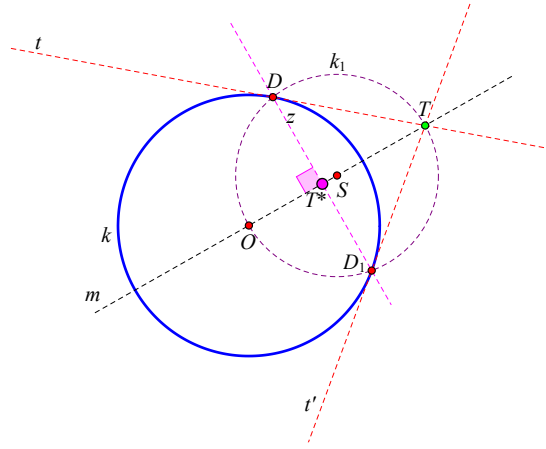


Figure 3: Construction of the inverse image of a point located outside the inversion circle.

Construction steps:

1. Draw a line m through points O and T .
2. Construct the tangent t from point T to the circle $k(O, r)$.
3. Let D be the point of contact of the tangent t .
4. At point D , draw a perpendicular z to the line m .
5. The line m and the perpendicular z intersect at the required point T^* , as shown in Figure 3. The same solution can be obtained by constructing the tangent t' through point D_1 . Now we need to prove that the constructed point satisfies the required conditions. Let us consider the right triangle $\triangle OTD$; then it follows

$$\frac{|OT^*|}{|OD|} = \frac{|OD|}{|OT|} \Rightarrow |OT| \cdot |OT^*| = |OD|^2.$$

Point D is on the inversion circle $k(O, r)$, i. e. $|OD| = r \Rightarrow |OT| \cdot |OT^*| = r^2$.

3 A BRIEF OVERVIEW OF *E*-GEOMETRY, *G*-GEOMETRY, AND *M*-GEOMETRY

3.1 Definition

The previous definition of inversion was geometric. However, for visualisation using Sketchpad, it is necessary to introduce the definitions of metric space, as well as the definitions of distance and “special” geometries.

A metric space is an ordered pair (H, d) , where H is a set and d is a metric on H , i.e. a function

- $d : H \times H \rightarrow R$ such that for each $x, y, z \in H$
- $d(x, y) \geq 0$,
- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, z) \leq d(x, y) + d(y, z)$.

The function d is also called the distance function or simply distance. If H is a set of points in a plane and d is a metric defined on it, then we have a metric space known as plane geometry.

If the metric is Euclidean, then we have Euclidean geometry. Distance is related to the Pythagorean theorem. In non-Euclidean geometry, the Pythagorean theorem does not hold, and distance is calculated differently.

The distance between two sets of points is the smallest possible distance (or infimum distance) between points in one set and the other. For our purposes, we will need different metrics. Let us consider them. Given the Cartesian rectangular coordinate system xOy and the points $T(x, y)$, $A(a, b)$ (see Figure 4).

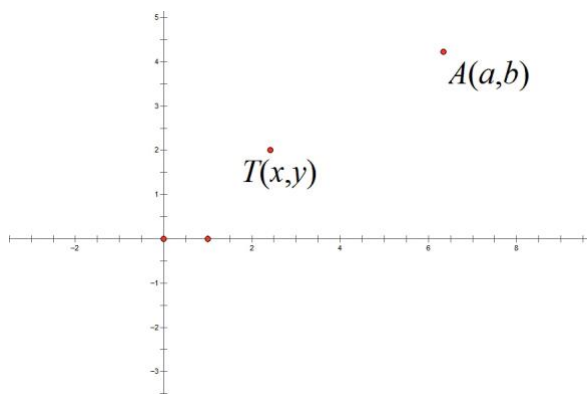


Figure 4: Points in the Cartesian rectangular coordinate system.

In Euclidean geometry, which we will call *E*-geometry, the distance between two points *T* and *A* is given by

$$|TA| = d_E(T, A) = \sqrt{(x - a)^2 + (y - b)^2}.$$

In taxicab geometry, a special case of Minkowski geometry, which we will call *M*-geometry, the distance between points *T* and *A* is given by

$$|TA| = d_M(T, A) = |x - a| + |y - b|.$$

In *E*-geometry, the shortest distance between two points is the length of the diagonal of a rectangle, or the hypotenuse of a right triangle, while in *M*-geometry it is the sum of the lengths of the lines parallel to the coordinate axes. Figure 5 shows the distance between two points in *E*-geometry and *M*-geometry.

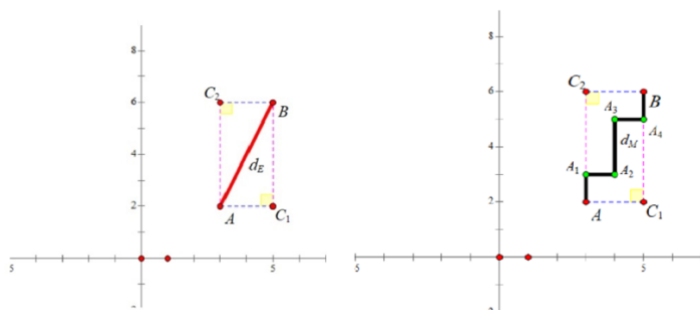


Figure 5: Distance between two points in *E*-geometry and *M*-geometry.

The analogy observed when considering the distance between points T and A in E -geometry and M -geometry suggests a definition for the distance between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ in P -geometry, i.e. it holds

$$|AB| = \begin{cases} |x_1 - x_2|^1 + |y_1 - y_2|^1 \dots \dots M - geometry \\ (|x_1 - x_2|^2 + |y_1 - y_2|^2)^{\frac{1}{2}} \dots \dots E - geometry \\ (|x_1 - x_2|^p + |y_1 - y_2|^p)^{\frac{1}{p}}, p \in \mathbb{R}_+ \dots P - geometry \end{cases}$$

In the Lamé equation of the unit trigonometric circle, $|x|^p + |y|^p = 1$, we “introduce” the parameters a and b by analogy with the unit circle $x^2 + y^2 = 1$ from Euclidean geometry, when the circle “transitions” into an ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

The result is the Lamé P -ellipse: $\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1, a, b \in \mathbb{R} - \{0\}, p > 0$. The parameter a is the major semi-axis, and b is the minor semi-axis.

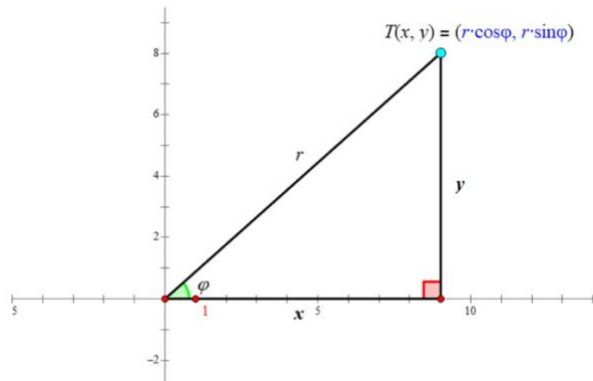


Figure 6: Connection between the representation of a point in the polar and Cartesian coordinate systems.

Let us write the Lamé P -ellipse in polar form. The “connection” between the Cartesian and polar coordinate systems is well known and given by $x = r \cos \varphi$, $y = r \sin \varphi$, as shown in Figure 6. By including those coordinates in

$$\left|\frac{x}{a}\right|^p + \left|\frac{y}{b}\right|^p = 1,$$

we obtain the following,

$$r^p \cdot \left(\left| \frac{\cos \varphi}{a} \right|^p + \left| \frac{\sin \varphi}{b} \right|^p \right) = 1$$

respectively

$$r^p = \left(\left| \frac{\cos \varphi}{a} \right|^p + \left| \frac{\sin \varphi}{b} \right|^p \right)^{-1}.$$

After exponentiating with $\frac{1}{p}$ the last expression, the trigonometric notation is obtained

$$r = \left(\left| \frac{\cos \varphi}{a} \right|^p + \left| \frac{\sin \varphi}{b} \right|^p \right)^{-\frac{1}{p}}.$$

A further step in the generalisation of the P -ellipse was made by the Belgian biologist and mathematician Johan Gielis (1962), who in 2003 formulated the so-called Gielis superformula. Today, this formula is more accurately referred to as the Gielis formula. Thus, G -geometry is born. Gielis further suggested that his formula (GF) could be added to, multiplied by, or compounded with some elementary function or composition of elementary functions $f(\theta)$, i.e., he defined $r = f(\theta) \cdot GF$.

To illustrate Gielis' idea, let us show different visualisations by changing various combinations of parameters and functions. Let the parameters be $m = 3$, $a = b = 1$, $n_1 = n_2 = n_3 = 1$, $n_4 = 0.20$, $k_1 = 7.13$; $m = 5$, $a = b = 1$, $n_1 = n_2 = n_3 = 1$, $n_4 = 0.20$, $k_1 = 7.13$; and $m = 10$, $a = b = 1$, $n_1 = n_2 = n_3 = 1$, $n_4 = 0.20$, $k_1 = 7.13$ and the function

$$f(\theta) = e^{\left(\left| \frac{1}{k_1} \cos\left(\frac{m}{2}\theta\right) \right|^{n_4} \right)}.$$

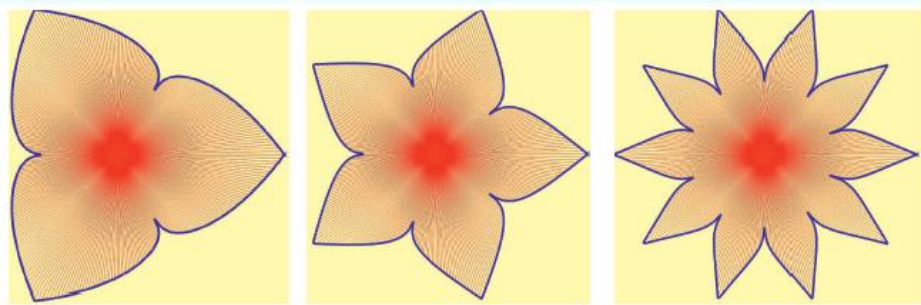


Figure 7: Gielis generalisation with exponential function.

By changing the parameters: $m = 4$, $a = b = 1$, $n_1 = n_2 = n_3 = n_4 = 1$, $k_1 = 0.2$, $k_2 = 1$ and $k_2 = 2$ and the function

$$f(\theta) = \ln \left(\left| \frac{1}{k_1} \cos \left(\frac{m}{k_2} \theta \right) \right|^{n_4} \right).$$

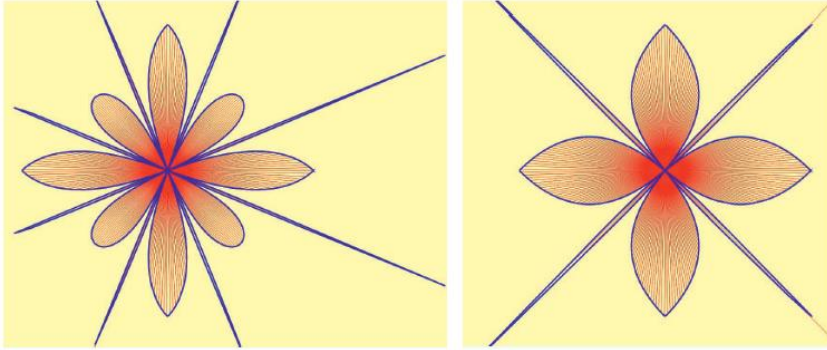


Figure 8: Gielis generalisation with logarithm function.

3.2 Sketchpad and its tools

This is the moment when The Geometer's Sketchpad proved to be a "lifesaver". For each point (or set of points – figure, segment, line, circle, arc, etc.) we can simply construct an inverse image.

a) Parameters/coefficients

- Geometric given parameters: Using The Geometer's Sketchpad, it is very easy to draw seven lines segments and associate them with decimal numbers representing their lengths. Moving the endpoints that define these segments changes their lengths, which in turn defines the values of all necessary parameters. By stretching a segment to the right, the parameter takes on positive values; by stretching it to the left, it takes on negative values.

- Numerically given parameters: Using The Geometer's Sketchpad, it is very easy to create seven numerical parameters. Using *Number | New Parameter*, we define these parameters. Their values can be changed or entered by double-clicking on the parameter. This opens the *Edit Parameter Value* window, where a specific value can be entered, resulting in the definition of the GF function.

Defining inversion

Using the *Transform | Define Custom Transform* command, we will define the transformation with which we will map objects.

1. Draw a point in the coordinate plane.
2. Construct the inverse image of that point.
3. Select the point and its image, and in *Transform | Define Custom Transform* in Sketchpad, define the inverse mapping.
4. Using this defined transformation, we can easily map any planar object by selecting the object and applying this transformation to it.

We obtain the object and its transformed image on the screen. By changing the parameters (coefficients), we achieve different “effects” of the object’s transformation, that is, its transformed image.

Inversion and the unit circle

For a given circle $k(O, r)$ with centre at point O (the origin of the coordinate system), we can define an inversion: the inversion of a point T in the plane is the point T' if certain conditions are met

$$|OT| \cdot |OT'| = r^2, T' \in \overrightarrow{OT}.$$

Figure 9 shows the construction of the inverse images of a point inside and outside the unit circle. Figure 10 shows the construction of the inverse image of a circle and a line with respect to the unit circle centred at the origin of the Cartesian coordinate system.

The construction was created using the Sketchpad programme.

Although it is possible to construct images of geometric figures classically, if the ordered pair of a point and its inverse image is known, the construction of the solution is straightforward using Sketchpad, that is, the special transformation Locus or geometric locus of points.

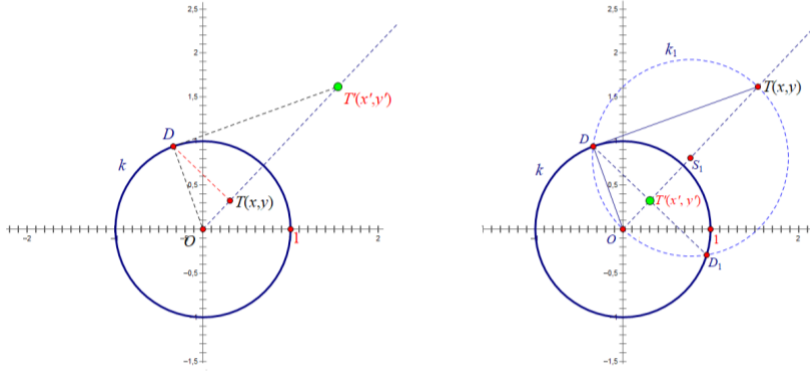


Figure 9: Construction of the inversion of a point inside or outside the unit circle.

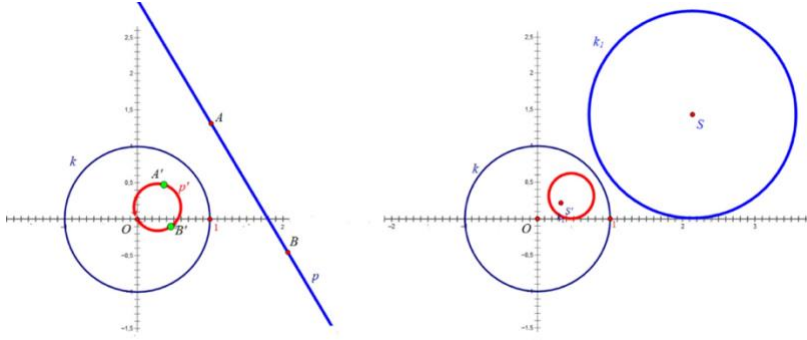


Figure 10: Construction of the inversion of a line and circle.

Let us see how the condition from the definition of inversion changes if the unit circle is given. Then, from the definition, it follows,

$$|OT'| = \frac{1}{|OT|}$$

or in E -geometry,

$$|OT'| = (|x|^2 + |y|^2)^{\frac{1}{2}}.$$

In M -geometry for the unit circle we get, $|OT'| = (|x| + |y|)^{-1}$ and in P -geometry

$$|OT'| = (|x|^p + |y|^p)^{\frac{1}{p}}.$$

In G-geometry we get

$$|OT'| = \left(\left| \frac{\cos(\frac{m}{4}\varphi)}{a} \right|^{n_2} + \left| \frac{\sin(\frac{m}{4}\varphi)}{b} \right|^{n_3} \right)^{\frac{1}{n_1}}, a, b, n_1 \in \mathbb{R} - \{0\}, n_2, n_3 \in \mathbb{R}.$$

The “external” exponents of inverse objects in these geometries are opposites. In Gielis’ formula, these exponents are -1 and 1, and the inversion is shown algebraically. As a result, it is possible to draw petals because inverses are obtained on flowers. Let us draw inverse images of a circle in different geometries. In the pictures, the objects are black, and their inverses are red. In Figure 11, the inverse circles in *E*-geometry are drawn.

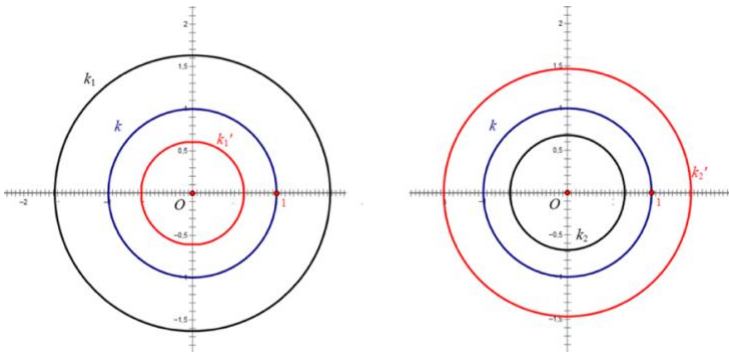


Figure 11: The inverse circles in *E*-geometry.

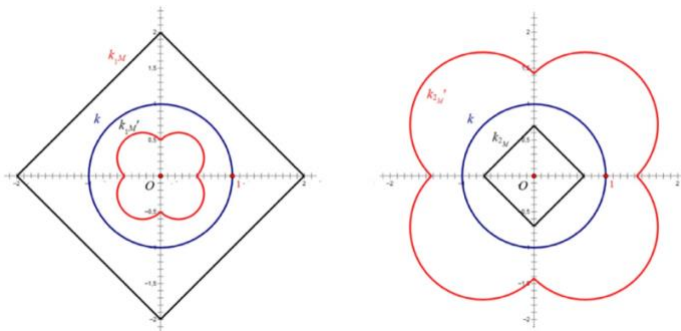


Figure 12: The inverse circles in *M*-geometry.

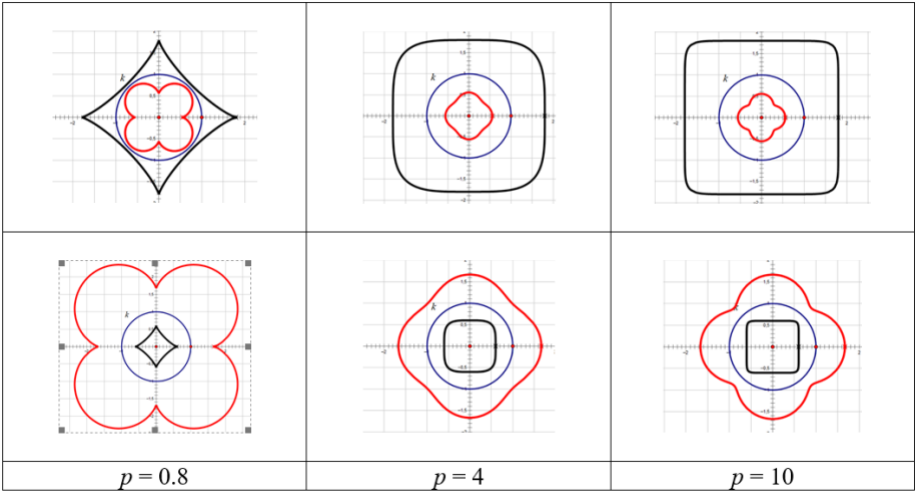


Figure 13: The inverse circles in P -geometry for different parameters.

5.VISUALISATION OF FLOWERS

Let us try to apply everything we have discussed so far to a few of Gielis’ “flowers” by drawing the petals.

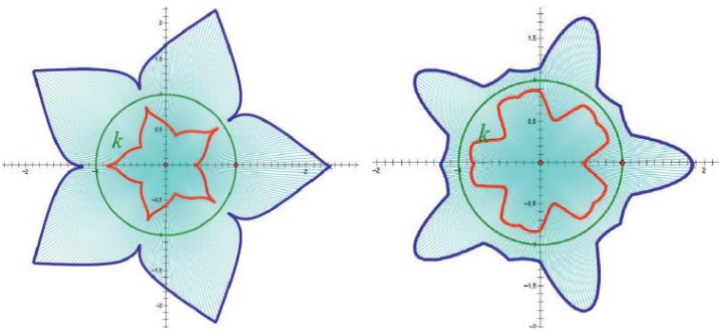


Figure 14: Two flowers in G -geometry.

In Figure 14, the flower petals are drawn with Sketchpad by changing the parameters as follows: $m = 5$, $a = b = 1$, $n_1 = n_2 = n_3 = 1$, $n_4 = 0.2$, $k_1 = 17.04$; or $m = 10$, $a = 1.5$, $b = 0.8$, $n_1 = 2$, $n_2 = 1$, $n_3 = 2$, $n_4 = 1$, $k_1 = 7.04$. Now, let us look at the flower in Figure 15a. It is a photograph of a flower of succulent *Huernia*, a flower whose large, fused petals and small sepals imitate the shape of a star.

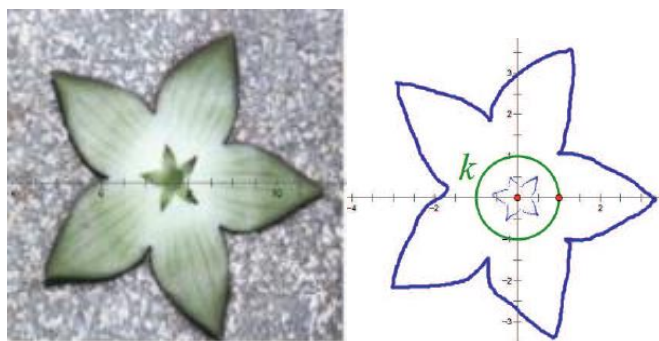


Figure 15: a) Abaxial (underside) image of *Huernia* flowers; b) the sketch of *Huernia*.

Let us sketch this flower and its inverse with the use of Sketchpad. In Figure 15b, the petals have their own inverse, which are the sepals, and vice versa. Figures 16 and 17 show the inverses of the flowers of the coastal thorn (Latin: *Pancratium maritimum*) and the autumn thorn (Latin: *Adonis autumnalis*), which are endangered plants in Croatia.



Figure 16: *Pancratium maritimum*.

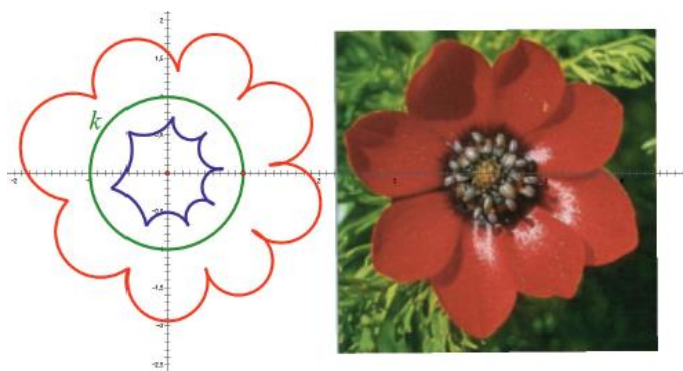


Figure 17: *Adonis autumnalis* and its inverse.

Figure 18 shows the flower of the Vršac mallow (lat. *Hibiscus trionum*) and its inverse drawn in Sketchpad for the parameters: $a = 4,07$, $b = 5,30$, $m = 10, 00$; $n_3 = - 2,15$, $n_2 = - 0,95$ i $n_1 = 9,52$.

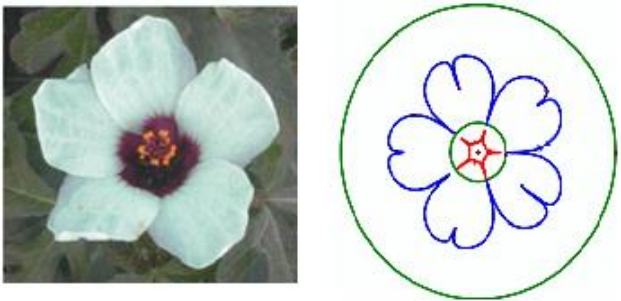


Figure 18: *Hibiscus trionum* and its inverse.

With the same parameters, different visualisations of the inverse representation are obtained. Changing the radius of the inversion circle will produce different representations of the same “object”.

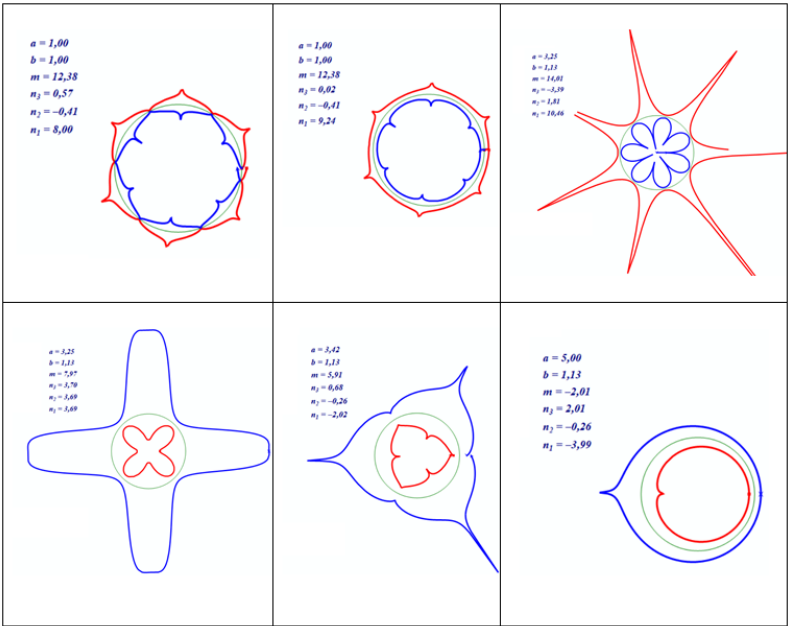


Figure 19: The blue objects are drawn using the Gielis formula, while the red ones (inversions) are drawn by swapping the exponents $\frac{1}{n_1}$ and $-\frac{1}{n_1}$, that is, the opposite exponent is used.

CONCLUSION

The study of inversion transformation in mathematics and geometry, along with the application of their properties, led to the visualisation of flowers and petals, or rather, to the consideration of familiar geometric and mathematical concepts in a new context: the world of flowers. Examples of visualisations of flowers and petals using Sketchpad are among the best and most beautiful illustrations of how mathematics and geometry are present in the plant world. With this powerful tool, any visualisation is possible.

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